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A discussion of the method of dimensionality reduction

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Abstract

The Method of Dimensionality Reduction (MDR) can be regarded as a formalism for analytical solution of some commonly encountered classes of contact problems using a "mechanical intuition" based on the Winkler foundation model. Such an approach makes it much easier to account for a wide range of physical effects associated with contact interaction (e.g. friction, adhesion, and damping). However, there is still a controversy about the method and its applications (see, e.g., the comment on validity of the MDR-based model of rough contact) – which we believe comes from a misunderstanding of the method itself, and which, in turn, can be reconsidered in view of the recently published book on the MDR. The MDR was originally introduced for Hertz's problem of axisymmetric frictionless local contact and was generalized subsequently for arbitrary axisymmetric geometry of linearly elastic bodies in unilateral local contact. The latter problem, for which the MDR yields the exact analytical solution, can be viewed as a base case that is used to extend, in a unified manner, the model of local contact by taking into account adhesion, friction, and viscous damping. In what follows, we overview the main concepts of the method starting with the base-case contact problem in which the MDR is rooted, and discuss limitations of the MDR as well. For the sake of their completeness, some criticisms that apply equally to conventional contact mechanics solutions are also considered. It is emphasized that the axisymmetric Hertz-type contact problems with a circular contact area constitute the proven range of validity of the MDR, while the extension of the method to other types of contact (e.g. axisymmetric with a multiply-connected contact area, non-axisymmetric) is a field ripe for research.

Keywords

Contact mechanics, friction, dimensionality reduction

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Base-case problem: Axisymmetric normal frictionless local contact

Consider two linearly elastic bodies (with Young's moduli E_1, E_2 and Poisson's ratios ν_1, ν_2) bounded by axisymmetric surfaces $z = f_1(r)$ and $z = -f_2(r)$, which are touching at a single point O chosen as the origin of the system of cylindrical coordinates (see Figure 1a). Correspondingly, the gap function (measured in the initial configuration) is given by

$$f(r) = f_1(r) + f_2(r)$$

In frictionless local contact of linearly elastic bodies, their elastic contacts are combined into one, and usually the effective elastic modulus, E^*, is introduced as follows:

$$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}$$

The Hertzian theory of local contact assumes that with regard to the question of determining the contact pressure between the elastic bodies, they are treated as elastic half-spaces. (This explains why the contact interaction is called local.) Therefore, under the additional assumption that the contact area, ω, is circular (with radius a), the contact pressure density, $p(r)$, should satisfy the following conditions of unilateral contact

$$p(r) \geqslant 0, \quad 0 \leqslant r < +\infty \tag{1}$$

$$p(r) > 0 \Rightarrow \frac{1}{\pi E^*} \iint_{\omega} \frac{p(\rho)\,\mathrm{d}S}{R} = d - f(r) \tag{2}$$

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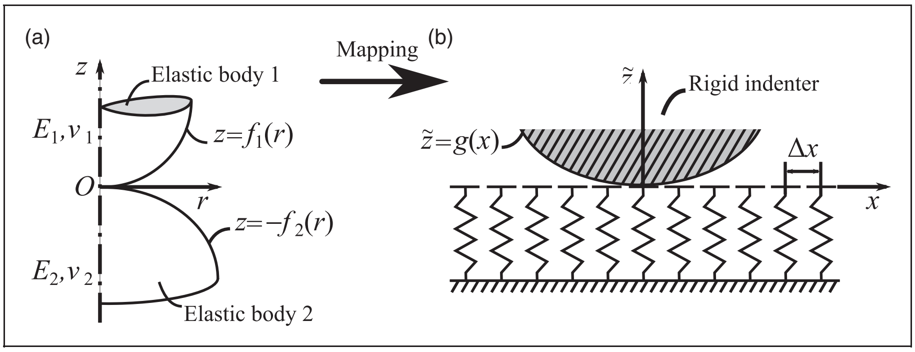

Figure 1. Initial contact configuration: (a) base-case contact problem and (b) MDR-based equivalent 1D contact model.

$$p(r) = 0 \Rightarrow \frac{1}{\pi E^*} \iint_{\omega} \frac{p(\rho)\mathrm{d}S}{R} \geqslant d - f(r) \tag{3}$$

Here, according to the solution of Boussinesq's problem, R is the distance between the point of observation (with the coordinate r) and the point of integration (with the radial coordinate ρ), while $\mathrm{d}S = \rho \mathrm{d}\rho \mathrm{d}\varphi$ is the area element in polar coordinates (ρ, φ).

The contact approach, d, of the contacting bodies, which enters the right-hand side of the governing integral equation (2) and the non-penetration condition (3), should be determined from the equilibrium equation

$$\iint_{\omega} p(\rho)\mathrm{d}S = F_N \tag{4}$$

Thus, the base-case problem of axisymmetric frictionless unilateral local contact consists of finding the contact pressure $p(r)$, the radius a of the contact area ω (where the contact pressure is positive), and the contact approach d, which for the known contact force F_N and the prescribed gap function $f(r)$ satisfy the relations (1)–(4).

MDR formalism for the base-case contact problem

Mapping into the 1D contact problem

First, the so-called discrete linear elastic foundation (see Figure 1b) is introduced as a system of independent, identical springs that are fixed to a rigid substrate and are separated from one another by a small distance Δx (called the discretization step). The stiffness of every individual spring element is

$$\Delta k_z = E^* \Delta x \tag{5}$$

where E^* is the effective elastic modulus for the contacting elastic bodies.

Second, the profile of the axially symmetric gap function $f(r)$, $r \in [0, +\infty)$ is replaced by an equivalent 1D profile $g(x)$, $x \in (-\infty, +\infty)$, for the equivalent rigid indenter according to the mapping rule

$$g(x) = |x| \int_0^{|x|} \frac{f'(r)\mathrm{d}r}{\sqrt{x^2 - r^2}} \tag{6}$$

For example, by the above Popov–Geike–Heß rule, the parabolic profile $f(r) = r^2/(2R)$ with the curvature radius R generates the equivalent 1D profile $g(x) = x^2/(2R_1)$, which is again parabolic but with the twice-smaller curvature radius $R_1 = R/2$. So, in this special case, the profile $z = f(r)$ is mapped into the profile $\tilde{z} = 2f(x)$, or, in other words, the original parabolic profile is stretched by a factor of 2 in the vertical direction.

Solving the 1D contact problem

When the equivalent rigid indenter is pressed into the discrete elastic foundation, its local indentation is given by

$$u_z(x) = d - g(x), \quad x \in (-a, a) \tag{7}$$

where the half-length of the contact interval a should be determined from the unilateral contact condition stating that the elastic foundation can produce only positive reaction.

Due to the applied deformation, an individual elastic spring element with a coordinate x produces the reaction force

$$\Delta f_N(x) = \Delta k_z \cdot u_z(x) \tag{8}$$

where the spring stiffness Δk_z is given by (5). Since, in unilateral contact, the spring reaction forces cannot take negative values, we have (cf. formula (1))

$$\Delta f_N(x) \geqslant 0, \quad x \in (-\infty, +\infty) \tag{9}$$

while inside the contact interval, we will have

$$\Delta f_N(x) > 0, \quad x \in (-a, a)$$

So, the formulas (7)–(9) determine the contact interval $(-a, a)$ in such a way that

$$g(a) = d \tag{10}$$

The total normal force needed to press the indenter against the elastic foundation is evaluated as the sum of all contributions of single springs $\Delta f_N(x)$ for $x \in (-a, +a)$, which, as $\Delta x \to 0$, reduces to the integral

$$F_N = E^* \int_{-a}^{a} (d - g(x))\mathrm{d}x \tag{11}$$

Finally, in view of (5) and (9), the 1D distributed foundation reaction, which is defined as $q(x) = \Delta f_N(x)/\Delta x$, is simply proportional to the local indentation $u_z(x)$, given by equation (7), with the proportionality constant E^*.

Inverse mapping (interpretation of the 1D contact model solution)

The 1D contact problem solved in the previous step reflects the main features of frictionless unilateral contact problems, but not all (for instance, the surface of the elastic foundation outside of the contact interval is left undeformed).

It is remarkable that the MDR establishes direct equivalence relations between the half-length of the 1D contact interval and the half-diameter of the original contact area (both characteristic sizes are denoted by the same symbol a) as well as between the normal contact forces denoted by F_N. However, the interpretation of the contact pressure density $p(r)$ in terms of the 1D elastic foundation reaction $q(x)$ is not so simple and requires the application of some integral transformation somewhat similar to that that appeared in formula (6). Nevertheless, for the base-case contact problem, the MDR yields the complete exact analytical solution (based on the contact pressure distribution, the elastic displacements outside the contact area can be recovered as well). It is to emphasize here that the MDR is based on the Galin–Sneddon solution[1,2] to the axisymmetric contact problem (1)–(3) (see also literatures[3–6]).

Normal frictionless local contact with adhesion

In Popov and Heß[7] (see Chapter 4), the Johnson–Kendall–Roberts (JKR) theory of adhesive contact [8] is represented in terms of the MDR, whose elegance and simplicity can be now fully appreciated. Indeed, the extension of the 1D base-case contact model, which accounts for adhesion, is based on the following requirement for the individual spring elements (cf. formula (9))

$$\Delta f_N(x) \geqslant -\Delta f_A, \quad x \in (-\infty, +\infty) \tag{12}$$

Here, Δf_A is the absolute value of the normal force required to break an adhesive contact with the spring element with a coordinate x. (Note that the convention of positive compressive spring reaction force is used.)

The quantity Δf_A is determined through the spring critical elongation, $\Delta l_{\max}$, by the obvious formula $\Delta f_A = \Delta k_z \cdot \Delta l_{\max}$, while the latter is obtained from the JKR theory. The important issue is that the quantity $\Delta l_{\max}$ depends on the radius, a, of the *a priori* unknown circular contact area, and this fact is denoted by $\Delta l_{\max}(a)$. It is interesting to note that the reason that $\Delta l_{\max}(a)$ is a function of a is that in the JKR theory, the contact force F_N is lower than that in the non-adhesive contact by an amount equal to the force on a circular flat punch of radius a that is just sufficient to generate a stress singularity with an energy-release rate equal to the interface energy.

Now, the condition of unilateral contact with adhesion (12) can be rewritten in the form

$$\Delta f_N(x) \geqslant -\Delta k_z \cdot \Delta l_{\max}(a), \quad x \in (-\infty, +\infty) \tag{13}$$

Correspondingly, taking into account formula (7) for the local indentation along with the elastic spring constitutive equation (8), we arrive at the following equation for the contact radius (cf. formula (10))

$$g(a) - \Delta l_{\max}(a) = d \tag{14}$$

At the same time, the force–displacement relationship (connecting the contact force F_N and the contact approach d) is given by the same formula (11).

Tangential local contact with friction

Let us consider again the undeformed configuration of two elastic bodies in local contact shown in Figure 1(a), but now assuming frictional contact according to Coulomb's law. Under the action of external normal loading F_N, the two bodies will interact with each other without tangential stresses at the contact interface, if and only if they are elastically similar, i.e. when

$$\frac{1 - 2\nu_1}{G_1} = \frac{1 - 2\nu_2}{G_2} \tag{15}$$

where G_1 and G_2 are the shear moduli of the contacting elastic bodies.

However, any subsequent application of tangential loading will give rise to tangential stresses, and a ring-shaped slip region will develop at the edge of the circular contact area, while inside the contact area, the frictional forces prevent slipping. These phenomena can be described analytically in the framework of Cattaneo–Mindlin theory,[9,10] and the radius of the stick region can be evaluated along with the tangential stresses and the relative displacements in the slip region.

It is interesting and of paramount practical significance that for the loading protocol specified above, the MDR provides the exact one-to-one mapping of tangential contact with partial slip for arbitrary axisymmetric elastically similar bodies (see Popov and Heß,[7] Chapters 5 and 18).

The extension of the MDR formalism to the case of frictional contact requires to equip the discrete elastic foundation with the spring tangential stiffness

$$\Delta k_x = G^* \Delta x$$

where G^* is the effective elastic shear modulus for the contacting elastic bodies given by

$$\frac{1}{G^*} = \frac{2-\nu_1}{4G_1} + \frac{2-\nu_2}{4G_2}$$

Furthermore, the MDR formalism was adopted in Chapter 6 to deal with rolling contact, which manifests itself in a number of important engineering applications. This allows to substantially simplify the treatment of complex contact problems without digressing into mathematical details. At the same time, it must be remembered that, being approximate, the Cattaneo–Mindlin theory is only strictly applicable to elastically similar bodies in local contact, or at least to the case when Dundurs' constant

$$\beta = \frac{(1-2\nu_1)/G_1 - (1-2\nu_2)/G_2}{(1-2\nu_1)/G_1 + (1-2\nu_2)/G_2}$$

is equal to zero (see equation (15)), and the same holds for the results obtained by help of the MDR.

It should be emphasized here that the Cattaneo–Mindlin solution for static contact is generally approximate because the predicted slip direction deviates from the assumed frictional stresses at points near the stick-slip boundary. This error was evaluated by Munisamy et al.[11] and found to be very small, and the theory of local tangential contact is exact in the special case for components exhibiting no Poisson's effect. By contrast, when a similar approach is applied to the problem of rolling spheres, the approximation becomes less accurate.

MDR as a three-step algorithm for a wide class of contact problems

Any application of MDR to a given contact problem consists of the following three steps: 1) mapping (of the original contact problem into the equivalent 1D problem for the discrete linear elastic foundation); 2) solving 1D contact problem; and 3) inverse mapping (interpretation of the results obtained on the previous step).

On the first step, the contact problem data (contact geometry and material parameters) are used to evaluate the input data of the corresponding 1D contact problem for the special discrete linear elastic foundation. This is achieved by means of the appropriate MDR transformations called the mapping rules. On the second step, the 1D contact problem for the equivalent rigid indenter is solved using the prescribed loading protocol. On the third step, the obtained solution of the equivalent 1D contact problem is employed for producing the solution to the original contact problem. This procedure requires application of the corresponding inverse mapping rule.

It should be noted that if the MDR were to be a method for complete solution of the contact problem, we would need the three-step algorithm described above, and in particular, we would need to be able to map back into the three-dimensional domain to find the contact pressure and the elastic displacements outside the contact area. It is indeed possible to do all three steps for the base-case axisymmetric contact problem, though the algebra thus involved is not significantly easier than in a conventional general solution.[1,2] However, in the case of rough contact (see the next section), where the MDR provides only an approximate solution, for example, the first two steps of the algorithm (mapping and solving) and only some elements of the third step (interpretation) are the focus of the MDR application, with the end of finding $F_N(d)$ and (possibly to some extent) the contact area.

MDR-based model of normal contact for rough surfaces

So far, the validity of the MDR is reduced to the question of the validity of the original theories of elastic local contact. It should be emphasized that the comment[12] (see also[13–16]]), in fact, addresses the validity of the MDR-based model of contact of elastic bodies with rough surfaces. Since the latter problem is intrinsically three dimensional, the mapping rules established in the axisymmetric case may not be applied, and the MDR approach (as it has been developed in Popov and Heß[7]) to the problem of rough contact uses profound ideas lying at the root of the MDR methodology.

The analysis in Popov and Heß[7] is based on the property of *self-affinity* for rough surfaces (that is, the invariance with respect to transformations $x' \to \psi x$, $y' \to \psi y$, $z' \to \psi^H z$ manifested by fractal surfaces with the Hurst exponent H when scaled in the spatial directions with an arbitrary magnification ψ) and the *self-similarity* of elastic solutions to the base-case contact problem. The latter feature of the contact problem solutions is grounded in distinctive aspects of the harmonic potentials in terms of which the contact problem governing relations (2) and (3) are formulated. This fact is well illustrated in Popov and Heß[7] (see Chapter 8) where the problem of heat transfer in local contact is worked out using the same MDR formalism as that developed for the base-case contact problem.

To illustrate the MDR-based model of rough contact, we consider the normal incremental stiffness

$$k_z = \frac{C}{H+1} L E^* \left(\frac{F_N}{L h E^*} \right)^{\frac{1}{H+1}} \tag{16}$$

where L is the characteristic length of the contact system, h is the root mean square of the roughness, and $C = 1.9412$ is a dimensionless constant. Equation (16) follows from formula (10.30), provided formulas (10.25) and (10.29) are taken into account (see Popov and Heß[7], Section 10). It is to note here that, in contrast to the base-case problem, in the case of rough contact problem, due to its essential complexity, no exact mapping rule has been established so far. That is why, the analytical formulas presented in Chapter 10 and, in particular, equation (16) written out above, should be viewed as a partially phenomenological model for normal contact of elastic rough surfaces. This means that, in its range of validity, the developed MDR-based model adequately predicts the main features of normal contact for randomly rough, statistically isotropic surfaces, whereas the inverse mapping (complete interpretation) of the solution to the 1D contact problem for the equivalent 1D rough profile and the discrete elastic foundation still requires further development and rigorous justification.

Note, by the way, that the constant C in equation (16) appears to be a product of the normalization constant $C_1 = 1.1419$ (for the stiffness k_z in the so-called state of complete contact) and another constant $C_2 = 1.7$ obtained from numerical data based on the one-dimensional indentation process for the equivalent 1D rough profile. Therefore, the prefactor C in formula (16) can be evaluated without reference to the complete contact state, which brings about a controversy with the application of the MDR to non-symmetric contact systems.

It is to underline that, generally speaking (that is abstracting from the MDR framework), the form of equation (16) assumes the contact of nominally flat rough surfaces. In the case of the contact of fractal surfaces without long wavelength cut-off, one of the critical features is that the long wavelength components leads to "clustering" of asperity contacts, and the resulting asperity interaction becomes an important contribution to the incremental stiffness when approaching the saturation level.

Nevertheless, it is very important to underline that the MDR-based model of rough contact does not employ any fitting parameters that need to be determined by nature experiments on the contact interaction between the real rough surfaces. The conversion factor $\lambda(H)$ for generating the equivalent 1D rough profile from a given randomly rough surface depends on its Hurst exponent H and can be taken to be π according the rule of Geike, or determined empirically (see Popov and Heß,[7] Section 10.7) by means of numerical simulations for both the three-dimensional and one-dimensional cases based on the criterion of preserving the contact stiffness in the MDR transformations.

Limitations of the MDR

Now, let us make some important comments on the MDR application procedure. First of all, it goes without saying that the solutions obtained via application of the MDR should be *a priori* considered as *approximate*, even though the MDR gives exact solutions (like in the base-case contact problem). Indeed, the MDR is based on the infinitesimal strain theory, linear constitutive relations, and the superposition principle, and that is why its results are quantitatively approximate in such contact problems as tribological ones. Moreover, some additional assumptions can be introduced to facilitate the use of the MDR (for instance, the evaluation of the frictional force between a rigid indenter and an elastomer in Section 11.3 Popov and Heß[7] neglects the inertia effects) that, generally speaking, may lead to a *qualitatively* different outcomes from those of a more rigorous analysis.

The validity of the MDR stems from the established strict equivalence between the axisymmetric Hertz-type contact problem and the corresponding 1D contact problem for a Winkler elastic foundation. It is worthwhile to emphasize that even in the axisymmetric setting, the MDR works only for singly connected (disk) contacts. For example, the MDR would not apply rigorously to an annular contact, where the direct application of the profile mapping rule (6) to a non-monotonic gap function $f(r)$ may even yield negative values for the equivalent 1D profile $g(x)$.

In complicated contact problems (including multiple physical effects, e.g., combinations of elasticity, energy dissipation, friction, and adhesion), the loading protocol plays a very important role and, generally speaking, a pair of special direct and inverse mapping rules should be formulated for a given class of loading protocols. For example, the mapping rule of Popov in the case of frictionless unilateral contact of viscoelastic bodies is *a priori* applicable only for monotonic loadings, when the contact area is not decreasing in time, because the last condition is a necessary requirement in the Lee–Radok elastic–viscoelastic correspondence principle.[17] That is to say that the extension of the MDR formalism for non-monotonic loadings requires a special consideration and justification.

The MDR deals with mathematical models of unilateral contact, and one can say that the method maps each of them into some mathematical model of 1D contact for a generalized discrete linear elastic foundation. If so, special attention should be paid to the validity of the assumptions (hypotheses) underlying the mathematical model for the contact problem under consideration. For instance, the Hertzian assumption of approximating the elastic bodies in local contact by elastic half-spaces implies that the MDR-based 1D mathematical model for the problem of frictionless contact between an elastic layer and an axisymmetric indenter does not account for the thickness effect even though the elastic layer is represented (through the MDR transformation) by a system of elastic springs of some finite length.

Needless to say that suggesting a mapping rule, one need to take care of designing the corresponding inverse mapping rule. In the MDR, the inverse mapping rule literally means an interpretation of the predictions drawn from the MDR-based equivalent 1D contact model in terms of the original contact problem. For example, in the base-case problem, the half-size of the 1D contact interval straightforwardly yields the exact value of the contact radius, while evaluating the contact pressures across the circular contact area requires a more elaborate inverse mapping. The matter is even worse in the MDR-based 1D contact model for rough contact, where the interpretation of the discrete 1D contact zone is still an open question. In particular, the previously suggested *approximate* inverse mapping for the area of contact lacks accuracy in the case of saturated contact close to the full contact situation (see the discussion in the literatures[12–16]). However, the circumstance that the accuracy of this particular approximate inverse mapping rule for the contact area drastically decreases with increasing size of the contact zone should not compromise the whole MDR, and it only means that a new more sophisticated inverse mapping rule is needed for the contact area interpretation.

In addition to the previous comment, we would like to note that the MDR is still far from its completion, and, in particular, the extension of the MDR for non-axisymmetric three-dimensional self-similar geometries is still an open question (see Popov and Heß,[7] Section 3.5). It is clear that the MDR formalism for local contact can be further developed and, for instance, generalized for transversally isotropic elastic bodies with the plane of isotropy parallel to the contact interface (this question is touched upon in Popov and Heß[7], Problem 7, Section 3.6).

Conclusion

Overall, the MDR represents a unified approach and a power tool for dealing with a broad spectrum of contact problems in engineering practice. Its field of application is as wide as that of the base-case contact problem and its generalizations discussed above. The recently published book Popov and Heß[7] is illustrated by a number of non-trivial examples, when the MDR can be especially useful to get insight into multi-scale contact and friction phenomena (e.g. Chapter 14 in Popov and Heß[7] considers the phenomenon of acoustic emission in rolling contacts, which is still an under-researched issue). At the same time, the versatility of the MDR must not be overestimated, and in the ambiguous situations the MDR-based results should be verified by a much more tedious finite element analysis, or otherwise the contact model assumptions should be reassessed. On the other hand, the MDR should not be regarded as a general purpose numerical method for contact problems. The practical significance of the MDR is that its development strives to establish the continuum mechanics-based link between the macro- and micro-scales in contact mechanics and friction.

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References

1. LA Galin and GML Gladwell (eds) *Contact problems: The legacy of L.A. Galin*. Dordrecht: Springer, 2008.
2. Sneddon IN. The relation between load and penetration in the axisymmetric boussinesq problem for a punch of arbitrary profile. *Int J Eng Sci* 1965; 3: 47–57.
3. Leonov MY. On the calculation of elastic foundations. *Prikl Mat Mekh* 1939; 3: 53–78.
4. Schubert G. Zur Frage der Druckverteilung unter elastisch gelagerten Tragwerken. *Ing-Arch* 1942; 13: 132–147.
5. Shtaerman IY. *Contact problems of elasticity*. Moscow, Leningrad: Gostekhteoretizdat, 1949.
6. Argatov II and Dmitriev NN. *Fundamentals of the theory of elastic discrete contact*. St. Petersburg: Polytechnics, 2003.
7. Popov VL and Heß M. *Method of dimensionality reduction in contact mechanics and friction*. Berlin, Heidelberg: Springer-Verlag, 2015.
8. Johnson KL, Kendall K and Roberts AD. Surface energy and the contact of elastic solids. *Proc R Soc Lond* 1971; A324: 301–313.
9. Cattaneo C. Sul contatto di due corpi elastici: distribuzione locale degli sforzi. *Rend delle' Accademia Naz dei Lincei* 1938; 27, Fasc. 7: 342–348, Fasc. 9: 434–436, Fasc. 10: 474–478.
10. MindIin RD. Compliance of elastic bodies in contact. *Trans ASME J Appl Mech* 1949; 16: 259–268.
11. Munisamy RL, Hills DA and Nowell D. Static axisymmetrical Hertzian contacts subject to shearing forces. *ASME J Appl Mech* 1994; 61: 278–283.
12. Persson BNJ. Contact mechanics for randomly rough surfaces: On the validity of the method of reduction of dimensionality. *Tribol Lett* 2015. DOI: 10.1007/s11249-015-0498-1.
13. Lyashenko IA, Pastewka L and Persson BNJ. On the validity of the method of reduction of dimensionality: area of contact, average interfacial separation and contact stiffness. *Tribol Lett* 2013; 52: 223–229.
14. Lyashenko IA, Pastewka L and Persson BNJ. Comment on Friction between a viscoelastic body and

a rigid surface with random self-affine roughness. *Phys Rev Lett* 2013; 111: 189401.
15. Li Q, Popov M, Dimaki A, et al. Reply. *Phys Rev Lett* 2013; 111: 189402.
16. Popov VL. Method of reduction of dimensionality in contact and friction mechanics: A linkage between micro and macro scales. *Friction* 2013; 1: 41–62.
17. Lee EH and Radok JRM. The contact problem for viscoelastic bodies. *J Appl Mech* 1960; 27: 438–444.

Author's closure on "A discussion of the method of dimensionality reduction by Ivan Argatov"

by V.L. Popov

I basically agree with the view presented in the paper by Argatov. In particular, it is clear that the MDR only provides an another interpretation for the results of the well-known theories of non-adhesive and adhesive normal contact and for tangential contact of axis-symmetrical bodies as well as for contact with linearly elastic elastomers. These classical solutions are associated with the names of Green, Collins, Sneddon (normal contact), Cattaneo, Mindlin, Jaeger, Ciavarella (tangential contact), Borodich, Yao, Gao (adhesive contact) as well as Lee and Radok (contact with viscoelastic media). It provides a simple intuitive picture for solution of all above-mentioned classes of contact problems. I also agree with the idea to stress once more that the contact with elastic foundation is only an intermediate step of computation in the framework of the MDR. The considered elastic foundation is only a formal mathematical object which should not be over interpreted in the sense of real form of interaction in the original system. This point is in reality the central point of the most confusion about the MDR. Our observation shows that many readers who see the picture of an elastic foundation start thinking that it is this foundation which is the object of consideration. This is of course completely wrong: The true object of consideration is the three-dimensional contact, and the elastic foundation is only an intermediate step of calculation. In this sense, I find that it is a good idea to stress, that the calculation in the framework of the MDR has to be completed with the third step of "interpretation" (or inverse transformation).

However, the paper of Argatov also touches upon some statements which I do not agree with. The main confusion in the current criticism of MDR (including the mentioned paper by Persson[12]) is connected with the notion of "validity" of a method. This notion can only be defined conditionally. The MDR can be valid and not-valid at the same time depending on what quantities we consider. I would like to discuss two properties – the *force–displacement relations* and the *contact area.*

- ***Force–displacement relations:*** Let us consider a non-adhesive contact of elastic half-space with another body of *arbitrary* topography. For this contact, the normal force F_N will be *some* monotonous function of the indentation d, that is $F_N = F_N(d)$. It is clear that it is *always possible* to construct such a one-dimensional profile $g(x)$ which will provide *exactly* the given dependence of the normal force on the indentation depth. The only prerequisite for this possibility is the monotonous dependence of the force on the indentation depth which is given for any non-adhesive contact. Thus, *with regard to the force–indentation relation*, the MDR will be *always* and *exactly* applicable independently of the form of the original topography. The only difference between different topographies is not the principal applicability or non-applicability, but the *availability of the rule* for calculation of the effective one-dimensional profile. In the case of axially-symmetric profiles, this rule is given by the equations of Galin–Green–Sneddon, while for non-axially-symmetric profiles, the exact form of the transformation is not known yet. However, there exist an approximate transformation for the case of self-affine randomly rough surfaces which was analytically derived by consideration of self-similarity of the contact problem and validated numerically. For any other topographies than these two classes, the MDR – with regard to the force-displacement relations – is of course still applicable, but the transformation rule is not discovered to this moment. On the other hand, there is a number of contact problems which can be reduced to the normal contact problem. These are: tangential contact and normal contact with linearly viscoelastic bodies. "Can be reduced" means in particular that if two systems have the same relation $F_N(d)$ for the normal elastic contact, then they will have the same behavior in tangential contact and in normal contact with a viscoelatic medium. The above-said means that all mentioned classes of contact problems can be solved with MDR for *absolutely arbitrary* profiles (not necessarily axis-symmetric), provided the effective one-dimensional profile is known (or that the normal contact problem has been solved).
- ***Contact area:*** Completely different situation occurs in the case of the contact area. At least for the above-mentioned approximate rule for the generating of equivalent profile for rough surfaces, the contact area cannot be described correctly in the framework of the MDR, the reasons for which are described in detail in the book.[7]

In conclusion, I would like to stress that the notion "validity" of the MDR is basically senseless as it must be completed with the restriction of the quantities which have to be described. This feature is not special for the MDR, but is implicitly assumed for any theory. The MDR is valid for any non-adhesive contacts independently of the topography, for any normal contacts with linear elastic viscoelastic media, and for any tangential contacts (within typical assumptions of the theories of Hertz or Mindlin as half-space-approximation, elastic similar media and so on). It is not valid for the contact area (at least with the presently available approximate rule for generating equivalent profile for rough surfaces).