ISSN 1029-9599, Physical Mesomechanics, 2014, Vol. 17, No. 1, pp. 50–57. © Pleiades Publishing, Ltd., 2014.
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Method of Dimensionality Reduction in Contact Mechanics and Tribology. Heterogeneous Media

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Received June 3, 2013

Abstract—The method of dimensionality reduction in contact mechanics is based on the mapping of three-dimensional contact problems onto one-dimensional systems. The method was developed and verified for homogeneous media. The present paper discusses the possibilities of generalizing the method to heterogeneous media with an illustration of basic ideas on the example of an elastic medium with a coating. The proposed method is easy to extend to media with lateral heterogeneity or to arbitrary heterogeneous media; this is illustrated by considering linear non-dissipative media. The proposed method can be used for calculation of contact and frictional interactions and for interpretation of experiments on indentation of heterogeneous media.

DOI: 10.1134/S1029959914010068

Keywords: contact, indentation, coatings, dimensionality reduction, heterogeneous media, adhesion

1. INTRODUCTION

Despite the rapid development of numerical simulation methods, tribology remains a branch of science and engineering in which numerical simulation methods still do not play a significant part. This is first of all due to the multiscale nature of friction processes and, consequently, due to the necessity to account for all spatial scales starting from the tribological system as a whole to the atomic scale [1–3]. Trends in the development of numerical methods in tribology and the prospects for hybrid methods that relate different scale levels are analyzed and discussed in paper [4]. For example, material wear and its transport from the friction zone can be described with the use of stochastic differential equations [5], whose parameters can be determined experimentally or by simulation with the method of movable cellular automata [6] or molecular dynamics method [7]. It seems crucial for the synthesis of different-scale models to develop data compression methods for compacted data transfer from one scale level to another as it is done in well-known methods of image coding or large database organization, such as Google Earth for example. Paper [4] suggests that the generalizing method in contact interaction mechanics and tribology can be the method of dimensionality reduction which is based on the mapping of three-dimensional contact problems onto one-dimensional. The method was later developed and verified for axially symmetric contacts [8] and for contacts with stochastic roughness [9–11]. It was shown that the method of dimensionality reduction can be applied to contacts with adhesion [8], viscoelastic media [12] and dynamic contact problems. The method of dimensionality reduction used to simulate static friction force in contacts oscillating with low amplitude [13] provides a very accurate description of experimental results [14]. Paper [15] demonstrates that the given method can be integrated into a macroscopic model so that system dynamics and contact forces are calculated immediately on each time step of numerical simulation. The theory of the method and its applications are overviewed in [16], and detailed proofs can be found in monograph [17].

Up to the present, the method of dimensionality reduction has been developed as applied to homogeneous media. However, many friction materials are heterogeneous media. The heterogeneity often lies at the heart of their functional properties. An example is the brake pad material which usually consists of a polymer matrix with solid inclusions. The present paper discusses how such materials can be described within the method of dimensionality reduction.

2. FORMULATION OF THE METHOD OF DIMENSIONALITY REDUCTION FOR HOMOGENEOUS MEDIA

Let us consider a normal contact between an elastic half-space with Young's modulus E and Poisson's ratio ν and an absolutely solid axially symmetric indenter. The shape of the indenter will be characterized by the function $z = f(r)$, where $r = \sqrt{x^2 + y^2}$ is the radius vector in the contact plane. The method of dimensionality reduction includes two steps [16]:

1. The elastic half-space is replaced with a one-dimensional chain of independent springs (Winkler foundation), as shown in Fig. 1, and the rigidity of each spring is determined according to

$$\Delta k_z = E^* \Delta x, \tag{1}$$

where

$$E^* = \frac{E}{1-\nu^2} = \frac{2G}{1-\nu}, \tag{2}$$

Δx is the distance between neighboring springs, and $G = E/2(1+\nu)$ is the shear modulus.

2. The profile $z = f(r)$ is substituted for the one-dimensional profile

$$g(x) = |x| \int_0^{|x|} \frac{f'(r)}{\sqrt{x^2 - r^2}} \mathrm{d}r, \tag{3}$$

where $f'(r)$ stands for the derivative.

The method of dimensionality reduction states that in the case of contact with modified profile (3) with the Winkler foundation determined according to Eq. (1), dependences between normal force F_n, contact radius a and indentation depth d will be exactly the same as in the initial three-dimensional contact problem. A rigorous proof of this statement was first provided in [8]. Notice that Steps 1 and 2 are independent of each other: the shape is transformed independently of properties of the

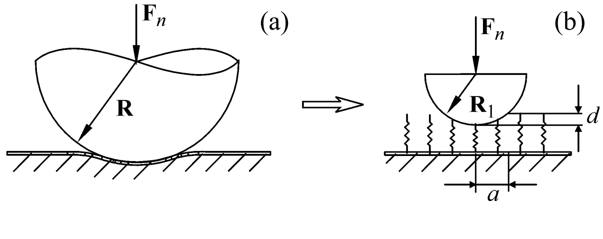

Fig. 1. Contact of a three-dimensional axially symmetric body with an elastic half-space (a) and its one-dimensional equivalent (b).

medium, and the replacement of the medium does not depend on the indenter shape.

The transformation is most simple in the case of profiles defined by the power function

$$f(r) = c_n r^n \rightarrow g(x) = \tilde{c}_n \, |x|^n, \tag{4}$$

$$\tilde{c}_n = \kappa_n c_n, \;\; \kappa_n = \frac{\sqrt{\pi}}{2} \frac{n\Gamma(n/2)}{\Gamma((n+1)/2)},$$

where $\Gamma(n)$ is the gamma function, i.e., the power profile is replaced by the function defined by a power function with the same power but "stretched" κ_n times in the vertical direction. For example, for a parabolic profile ($n = 2$) the scaling factor $\kappa_n = 2$, and for a conical indenter ($n = 1$) $\kappa_1 = \pi/2$.

Notice that the method of dimensionality reduction can also be used to calculate the distribution of normal stresses $\sigma_{zz}(r)$ in the initial three-dimensional problem. The stress calculation rule is the following: compression forces $\Delta f_n(x)$ of individual springs are determined in a one-dimensional problem. Then, the linear density of the force is found:

$$q(x) = \frac{\Delta f_n(x)}{\Delta x}. \tag{5}$$

Stress in the initial three-dimensional problem is given by the equation

$$\sigma_{zz}(r) = \frac{1}{\pi} \int_r^\infty \frac{q'(x)}{\sqrt{x^2 - r^2}} \mathrm{d}x. \tag{6}$$

The validation of this equation can be found elsewhere [8, 17].

Heß [8] showed that contacts with adhesion can also be rigorously described within the method of dimensionality reduction. In so doing, the following rule is introduced, apart from the above-described Steps 1 and 2: springs at the boundary of adhesive contact (in a one-dimensional equivalent model) are in indifferent equilibrium if their elongation Δl meets the condition

$$\Delta l = \Delta l_{\max}(a) = \sqrt{\frac{2a\pi\gamma_{12}}{E^*}}, \tag{7}$$

where γ_{12} is the effective surface energy of the interface (work of separation per unit area). Notice that in the case of adhesion we deal with a nonlocal separation criterion because ultimate elongation depends on the contact size. Emphasize, however, that individual springs of the Winkler foundation remain independent, and ultimate elongation depends only on the contact size, not on the indenter shape.

There are also similar rigorous theorems for tangential contacts [17]. In the present paper a normal contact problem is considered.

3. IS IT POSSIBLE TO DESCRIBE HETEROGENEOUS MEDIA WITHIN THE METHOD OF DIMENSIONALITY REDUCTION?

The possibility of describing axially symmetric contacts within the method of dimensionality reduction is governed by three main properties of contact problems.

A. In the case of indenting a rigid indenter into an arbitrary medium with linear rheology, the shape of its surface depends only on the indenter shape and indentation depth, not on the properties of the medium. Particularly, the surface shape will be the same for all homogeneous elastic media independently of their elastic properties, just as in indentation of fluids or viscoelastic bodies with an arbitrary linear rheology. The last property was first noticed in papers [18, 19]. A well-known example of this independence is the equation $a = \sqrt{Rd}$ that defines contact radius a in the Hertzian problem (contact between an elastic half-space and a parabolic indenter with curvature radius R). The contact radius obviously depends only on the indenter shape and indentation depth, not on the properties of the medium. This property holds not only for axially symmetric contacts but also for contacts of arbitrarily complex shape.

B. The differential rigidity of contact depends exclusively on the current configuration of the contact area, not on the history of its formation. This means that the differential rigidity of two bodies of different shape will be the same if they have the same contact areas in the given state. A proof of this statement can be found elsewhere [3, 20].

C. The contact rigidity of a round contact of radius a is equal to

$$k_z = 2aE^* \tag{8}$$

and hence proportional to the contact diameter.

It is easily seen that the above three properties make it possible to map three-dimensional media onto one-dimensional ones. Really, in contact with a one-dimensional series of independent springs the contact size depends only on the indenter shape and indentation depth, not on the rheological properties of individual springs. This property, however, is inherent in three-dimensional systems (Property A). Differential rigidity in a one-dimensional case may evidently depend only on the current configuration of the contact. This property is also inherent in an original three-dimensional system (Property B). Finally, when using springs with constant rigidity, the differential rigidity of contact in a one-dimensional case turns to be automatically proportional to the contact area size. This property is also inherent in the three-dimensional original (Property C). Owing to Properties A and B, the contact problem is divided into two independent parts: the definition of contact configuration does not depend on material properties, depending only on the shape of the body and indentation depth, while contact rigidity depends solely on the current configuration of the contact and does not depend on the body shape.

Property A is the most universal of the three and holds true independently of the shape of bodies and their structure. It holds equally for arbitrary heterogeneous media with linear rheology. A proof of the statement is a straightforward generalization of the corresponding proof for homogeneous media given in [3].

Properties B and C are not so universal. Their discussion is therefore crucial in the analysis of whether it is possible to generalize the method of dimensionality reduction to heterogeneous media.

Let us first discuss Property A as applied to heterogeneous media. By way of illustration, consider an elastic continuum with a coating of thickness h with elastic modulus E_0 and Poisson's ratio ν_0, which are different from the elastic properties (E_1 and ν_1) of the substrate material. We consider the indentation of a parabolic indenter into the continuum. It is obvious that at a small indentation depth the contact radius is defined exclusively by the coating deformation and is given by the Hertz formula $a = \sqrt{Rd}$. At a very large indentation depth it is defined by the deformation of the elastic substrate and is still given by the Hertz formula. At intermediate depths the Hertz formula is not guaranteed to hold true. However, in the majority of practically important cases, deviation from the Hertz formula will be relatively low. For example, paper [21] shows that deviation from the Hertz formula for media with a thin coating simulated as an elastic plate never exceeds 1.7%. In paper [22] the dependence between contact radius and indentation depth is studied for coatings with the elastic modulus ratio varying in the range from 10^{-2} to 10^2. Maximum deviation from results for a homogeneous medium increases with the growing elastic modulus ratio. For a softer coating at an elastic modulus ratio equal to 2 the maximum deviation of the indentation depth from the Hertz result at a given contact radius is about 10%. For rigid coatings on softer substrates maximum deviation is higher than for softer coatings on more rigid substrates. Thus, in the case of heterogeneous media, the independence of contact configuration on material parameters at a given indentation depth does not hold. However, the independence holds in limiting cases of very shallow or very deep indentation and can be used in the intermediate region as an approximation yielding qualitatively correct results. The above said about the medium with a coating is equally valid for laterally heterogeneous media. For example, for a matrix with exposed inclusions, the Hertz relation is valid both in indentation of each phase separately and at a relatively large contact radius when the material can be considered as homogeneous. In contact with two phases simultaneously or at a contact radius comparable to the size of inclusions, the Hertz relation is not rigorously valid but most often valid with rather good accuracy. In the present paper we use the formulated approximated independence of relations between indentation depth and contact radius, but nevertheless it is easy to go beyond this approximation. For example, papers [22, 23] present an analytical and numerical study of indentation of differently shaped bodies into an elastic medium with a coating to demonstrate that at a given contact radius the ratio between the real indentation depth d and the indentation depth d_{eq}, which would be obtained at the same contact radius in a homogeneous medium, depends only on the ratio of elastic constants and ratio of contact radius a to coating thickness h, rather than on the indenter shape:

$$\frac{d}{d_{eq}} = \zeta\left(\frac{E_1}{E_0}, \frac{a}{h}\right). \tag{9}$$

The form of this function must be defined in analytical or numerical analysis. As has already been said, we will not make such correction in the present paper, assuming that $\zeta = 1$.

Now turn to the discussion of Property C. Since according to Property B contact rigidity depends only on contact radius, then it is always possible to introduce an effective elastic modulus depending on contact size, so that the following relation would be formally satisfied:

$$k_z = 2aE^*_{eff}(a). \tag{10}$$

This approach was put forward in the context of studying hardness of materials. The following empirical dependence was proposed in [24] for the effective elastic modulus:

$$\frac{1}{E^*_{eff}} = \frac{1-\nu_0}{2G_0}(1-e^{-\alpha h/a}) + \frac{1-\nu_1}{2G_1}e^{-\alpha h/a}, \tag{11}$$

where α is a constant of the order of unity. The given dependence has no good theoretical justification, simply being an empirical interpolation between two limiting values of the elastic modulus, with transition occurring at $a \approx h$. Equation (11) interpolates elastic moduli under the assumption that corresponding rigidities are connected in series, which seems natural at first sight having regard to the layered structure of the system. Actually, interpolation on the assumption that rigidities are connected in parallel is more accurate. Paper [25] gives an exact expression for the effective elastic modulus of an arbitrary layered medium in the framework of perturbation theory and proposes simple interpolation formulas at a considerable difference of elastic coefficients. For a medium with a coating the equation is valid if

$$E^*_{eff} = \frac{1-\nu_{eff}}{2G_{eff}}, \tag{12}$$

where

$$\begin{aligned} \nu_{eff} &= \nu_1 + (\nu_0 - \nu_1)I_1(h/a), \\ G_{eff} &= G_1 + (G_0 - G_1)I_0(h/a), \end{aligned} \tag{13}$$

and the functions $I_1(\xi)$ and $I_0(\xi)$ are determined as follows:

$$I_1(\xi) = \frac{2}{\pi}\arctan\xi + \frac{\xi}{\pi}\ln\frac{1+\xi^2}{\xi^2}, \tag{14}$$

$$I_0(\xi) = \frac{2}{\pi}\arctan\xi + \frac{1}{2\pi(1-\nu)}\left[(1-2\nu)\xi\ln\frac{1+\xi^2}{\xi^2} - \frac{\xi}{1+\xi^2}\right]. \tag{15}$$

Both functions tend to zero at $\xi > 0$ and to unity at $\xi > \infty$. For a medium with an arbitrary dependence of elastic constants on depth, the following common expressions in the same approximation are derived in [25]:

$$\begin{aligned} \nu_{eff} &= \int_0^\infty \frac{dI_1(z/a)}{dz}\nu(z)dz, \\ G_{eff} &= \int_0^\infty \frac{dI_0(z/a)}{dz}G(z)dz. \end{aligned} \tag{16}$$

These expressions take an especially simple form in the case of incompressible media. Poisson's ratio in this case is equal to $\nu = 1/2$, and for the effective shear modulus we have

$$G_{eff} = \int_0^\infty \Phi(z/a)G(z)dz, \tag{17}$$

where

$$\Phi(\xi) = \frac{1+3\xi^2}{\pi(1+\xi^2)^2}. \tag{18}$$

It is obvious that integral (17) defines the elastic modulus value weighted with distribution (18). The approach of [25] is actually based on the assumption that the field of displacements in an elastic solid (including its surface) does not depend on material structure. If the assumption holds true, Property A (that contact configuration is independent of properties of the medium) is automatically valid.

A more general case of coatings with considerably different elastic coefficients is considered in [23] to show that deviation from relations of [25] can be strong. A more general interpolation formula can be derived from the following considerations. The effective shear modulus under wave-like surface deformation with wave vector q is given, at an arbitrary ratio of elastic coefficients, by the equation from [26]:

$$G_{\text{eff}} = G_0 \frac{(G_0 + G_1) - (G_0 - G_1)e^{-2qh}}{(G_0 + G_1) + (G_0 - G_1)e^{-2qh}}. \tag{19}$$

At indentation with a cylindrical indenter of radius a, we may assume for a rough estimate that $q = \pi/(2a)$:

$$G_{\text{eff}} \approx G_0 \frac{(G_0 + G_1) - (G_0 - G_1)e^{-\pi h/a}}{(G_0 + G_1) + (G_0 - G_1)e^{-\pi h/a}}. \tag{20}$$

This equation can be considered as a generalization and improvement of simple interpolation formulas (11), (13).

Inasmuch as equation (10) is valid, it is easy to formulate the mapping of a three-dimensional contact problem on a one-dimensional model. To do this, we must define a Winkler foundation in which the rigidity of springs is determined by the relation

$$\Delta k_z = E^*_{\text{eff}}(a)\Delta x. \tag{21}$$

Here, the rigidity of springs is a function of contact size. The situation is similar to that observed in description of adhesion.

4. EXAMPLES OF APPLYING THE METHOD OF DIMENSIONALITY REDUCTION TO INDENTATION OF LAYERED MEDIA

As has been already said, the method of dimensionality reduction can be formulated more rigorously with regard to the correction of Property A and in a more simple approximated form with no regard to the correction (in the approximation of [25]). There is no need to correct the contact shape if the difference between the elastic properties of the substrate and coating is insignificant. Below, we will restrict ourselves to this approximation. In the given case, the method of dimensionality reduction is formulated in the following way.

1. The elastic half-space is replaced with a one-dimensional series of independent springs (Winkler foundation); the rigidity of each spring is defined according to (21), where the effective elastic modulus is determined by Eqs. (12)–(15).

2. The profile $z = f(r)$ is replaced by a one-dimensional profile according to rule (3).

3. After indentation, contact rigidity is determined. To find force, it is necessary to perform indentation from the initial state without contact and to find force through integration. The last rule arises due to system nonlinearity related to the dependence of rigidity on contact size.

Let us illustrate the application of these rules by two examples.

4.1. Parabolic Indenter

Consider an indenter of shape $z_{3\text{D}} = r^2/(2R)$. According to Rule 2, this profile is replaced by the one-dimensional profile $z_{1\text{D}} = x^2/R$. Contact radius at indentation depth d is defined by the condition $z_{1\text{D}}(a) = d$ and is equal to $a = \sqrt{Rd}$ independently of properties of the indented medium. By considering incompressible media for simplicity $(\nu = 1/2)$, we find for differential rigidity

$$\frac{\mathrm{d}F_\text{n}}{\mathrm{d}d} = 2a \cdot 4G_{\text{eff}} = 8a\left[G_1 + (G_0 - G_1)I_0(h/a)\right]$$

$$= 8a\left[G_1 + (G_0 - G_1)\left(\frac{2}{\pi}\arctan\frac{h}{a} - \frac{h/a}{\pi(1 + (h/a)^2)}\right)\right]. \tag{22}$$

Using $\mathrm{d}d = (2a/R)\,\mathrm{d}a$, we rewrite the equation as

$$\frac{\mathrm{d}F_n}{\mathrm{d}a} = \frac{16a^2}{R}\left[G_1 + (G_0 - G_1)\right.$$

$$\left.\times\left[\frac{2}{\pi}\arctan\frac{h}{a} - \frac{h/a}{\pi(1 + (h/a)^2)}\right]\right]. \tag{23}$$

Integration yields the dependence of normal force on contact radius:

$$F_\text{n} = \frac{16a^3}{3R}\left[G_1 + (G_0 - G_1)\right.$$

$$\left.\times\frac{1}{2\pi}(4\arctan(h/a) + (h/a)^3\ln(1 + (a/h)^2)) - h/a\right]. \tag{24}$$

In the limiting cases, we have:

$$F_\text{n} = \frac{16a^3}{3R}\begin{cases} G_0, \; a/h \ll 1, \\ G_1 + (G_0 - G_1)3h/a, \; a/h \gg 1, \end{cases} \tag{25}$$

whence it follows that at small contact sizes the medium has properties of the coating, while at large contact sizes it has properties of the substrate. Approximation to the substrate properties is carried out by power rather than exponential law.

4.2. Conical Indenter

Equation (22) is valid independently of the indenter shape. In the case of a conical indenter, the only change is in the ratio between contact radius and indentation depth. Let the indenter shape is given by the equation $z_{3\text{D}} = r\tan\varphi$. According to the method of dimensionality reduction, the given profile is changed for the one-dimensional profile $z_{1\text{D}} = \pi/2\,|x|\tan\varphi$. Contact radius is defined by the condition $z_{1D}(a) = d$ and equals $a = (2/\pi)d/\tan\varphi$. Having regard to $\mathrm{d}d = \pi/2\tan\varphi\,\mathrm{d}a$, rewrite Eq. (22) as

$$\frac{\mathrm{d}F_\mathrm{n}}{\mathrm{d}a} = 4\pi\tan\varphi\, a\Bigg[G_1 + (G_0 - G_1) \times\left(2/\pi\arctan(h/a) - \frac{h/a}{\pi(1+(h/a)^2)}\right)\Bigg]. \quad (26)$$

Integration yields

$$F_\mathrm{n} = 2\pi a^2\tan\varphi\left[G_1 + (G_0 - G_1)\frac{2}{\pi}\arctan h/a\right]. \quad (27)$$

5. ADHESIVE FORCE IN A CONTACT WITH A COATED ELASTIC BODY

In the case of one-dimensional media, the condition of loss of adhesive contact is determined by condition (7). Its derivation is based on the consideration of balance between elastic energy of the deformed body and adhesion energy [27], or on the consideration of stress intensity factor at the adhesive contact boundary [28]. The latter method was used in [28] to find a general solution of an adhesion problem for arbitrary axially symmetric bodies. Since at a comparatively low material heterogeneity we assume that surface deformation is independent of material properties and rigidity is defined only by contact size, it is obviously the effective rigidity determined by Eq. (10) that appears in the energy balance. Surface energy is fully defined by coating properties. It is therefore necessary to substitute condition (7) in the method of dimensionality reduction for

$$\Delta l = \Delta l_{\max}(a) = \sqrt{\frac{2a\pi\gamma_{12}}{E^*_{\mathrm{eff}}(a)}}. \quad (28)$$

Let us illustrate this approach by the example of adhesive contact between a parabolic indenter and substrate with a coating.

As in the previous section, we replace the parabolic profile $z_{3D} = r^2/(2R)$ by the one-dimensional effective profile $z_{1D} = x^2/R$. At indentation depth d the vertical displacement of springs with coordinate x is equal to $u_z(x) = d - x^2/R$. Contact radius is defined by the equation $u_z(a) = -\Delta l_{\max}(a)$ or by

$$d = \frac{a^2}{R} - (2a\pi\gamma_{12})^{1/2}\Bigg[4\Bigg(G_1 + (G_0 - G_1) \times\left(2/\pi\arctan(h/a) - \frac{h/a}{\pi(1+(h/a)^2)}\right)\Bigg)\Bigg]^{-1/2}. \quad (29)$$

By introducing the dimensionless variables

$$\tilde{a} = a/a_c,\ \tilde{d} = d/d_c,\ \tau = G_0/G_1,\ \tilde{h} = h/a_c, \quad (30)$$

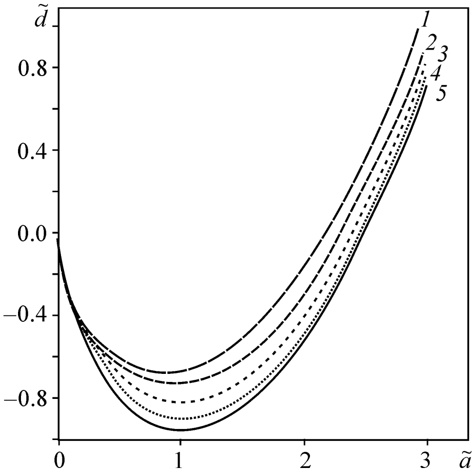

Fig. 2. Dependence of indentation depth on contact radius according to Eq. (32) at $\tau = 2$ and dimensionless layer thickness $\tilde{h}$, equal to 4 (*1*), 2 (*2*), 1 (*3*), 0.5 (*4*) and 0.25 (*5*).

where

$$a_c = \left(\frac{\pi\gamma_{12}R^2}{32G_1}\right)^{1/3},\ d_c = 3\left(\frac{\pi^2\gamma_{12}^2 R}{2^{10}G_1^2}\right)^{1/3}, \quad (31)$$

rewrite the equation in the form:

$$\tilde{d} = \frac{\tilde{a}^2}{3} - \frac{4}{3}\tilde{a}^{1/2}\Bigg[1 + (\tau - 1) \times\left(2/\pi\arctan(\tilde{h}/\tilde{a}) - \frac{\tilde{h}/\tilde{a}}{\pi(1+(\tilde{h}/\tilde{a})^2)}\right)\Bigg]^{-1/2}. \quad (32)$$

Dependence (32) for five values of dimensionless layer thickness is demonstrated in Fig. 2. The maximum negative value of indentation depth defines the point of loss of adhesive contact. The results are in good agreement with exact solutions obtained in [29]. A more detailed analysis of adhesive contacts will be carried out in a separate paper.

6. DESCRIPTION OF LATERALLY HETEROGENEOUS MEDIA

The method of determining effective parameters of a heterogeneous medium [25] is based on the observation that deformation of a three-dimensional medium slightly depends on its structure, at least in the case when the elastic properties of the medium are not highly heterogeneous. Owing to this property, deformation of a body of any given shape can be defined independently of properties of the medium. With known strain, energy of the medium can be calculated in a closed integral form and thus the effective properties of the medium can be formally defined. An absolutely similar approach can be applied to media with arbitrary heterogeneity, e.g., with layered structure directed perpendicularly to the contact surface or with the structure of a matrix with inclusions. In this case, the basis for the application of the method of dimensionality reduction is still equation (10), in which the effective elastic modulus will be the function of both the indenter radius and its position:

$$k_z = 2aE^*_{\text{eff}}(a, x, y). \quad (33)$$

In transition to a one-dimensional description, we must restrict ourselves to only one direction of motion along the two-dimensional surface, e.g., by assuming that $y = 0$. This will determine the "instantaneous value" of the effective elastic modulus in the system depending on the position and size of the contact: $k_z = 2aE^*_{\text{eff}}(a, x, 0)$. A detailed study of laterally heterogeneous media will be performed in a separate paper.

7. CONCLUSION

The paper considers general theoretical aspects of the applicability of the method of dimensionality reduction to the contact between axially symmetric bodies and heterogeneous media. Analysis of the literature on the indentation of layered media shows that the method can be applied to heterogeneous media on condition of introducing the dependence of effective profile and effective elasticity on contact size, as well as on coordinate in the case of laterally heterogeneous media. Besides, in the case of heterogeneous media, contact configuration is used directly to determine differential contact rigidity, not force. The latter must be calculated by integration. Despite the mentioned complication of the method, it nevertheless accelerates many times the calculation of contact forces because there is no need to solve the integral equation of the contact problem.

The approximation of [25] makes it possible to significantly simplify the problem: contact configuration is defined independently of properties of the medium, and the effective elastic modulus turns to be the only quantity depending on contact size.

The author is thankful to M. Heß and I. Argatov for the discussion of the paper and literature recommendations.

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