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# Simplified Simulation of Fretting Wear Using the Method of Dimensionality Reduction

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**Abstract**—We study the problem of wear of a rotationally symmetric profile subjected to oscillations with small amplitude. Under these conditions, sliding occurs at the boundary of the contact area while the inner parts of the contact area may still stick. In a recent paper, Dimaki with colleagues proposed a numerically exact simulation procedure based on the method of dimensionality reduction (MDR). This drastically reduced the simulation time compared with conventional finite element simulations. The proposed simulation procedure requires carrying out the direct and the inverse MDR transformations in each time step. This is the main time consuming operation in the proposed method. However, solutions obtained with this method showed a remarkable simplicity of the development of wear profiles in the MDR space. In the present paper, we utilize these results to formulate an approximate model, in which the wear is simulated directly in the one-dimensional space without using integral transformations. This speeds up the simulations of wear by further several orders of magnitude.

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## 1. INTRODUCTION

Fretting wear occurs in contacts subjected to oscillations with small amplitude. It is one of the causes for malfunctioning of engineering components and was studied intensively in connection with such applications as fretting of tubes in steam generators [1–3], medical applications [4], electrical contacts [5], fretting fatigue of dovetail blade roots [6, 7] and many others. In theoretical modeling of wear, very often an equation is used which states that the wear volume $\Delta V$ is proportional to the normal force $F_\text{n}$, the relative tangential displacement $u_x$ of the contacting bodies and inversely proportional to the hardness $\sigma_0$:

$$\Delta V = kF_\text{n}u_x/\sigma_0. \qquad (1)$$

This wear equation was suggested already in 1860 by Reye [8], and was later derived and experimentally justified for abrasive [9] and adhesive wear [10] (derivations see also in [11]). To describe the detailed shape changes due to wear, the wear law (1) is often formulated as local relation

$$\Delta h(x, y) = k\frac{p(x, y)u_x(x, y)}{\sigma_0}, \qquad (2)$$

where $\Delta h$ is the linear wear, $p(x, y)$ is the local pressure and $u_x(x, y)$ is the local relative displacement.

Application of the local rule (2) requires solving the contact problem for any current configuration. The main part of the literature on theoretical modeling of fretting wear is devoted to numerical solution of the contact problem using finite element or boundary element programs (see e.g. [12]) and implementation of the Reye–Archard–Khrushchov law in them. In the case of rotationally symmetric profiles, the simulation can be substantially speed up by solving the contact problem with the method of dimensionality reduction (MDR) [13] as it was done in [14]. The method of dimensionality reduction is based on the mapping of three-dimensional problems on the properly defined one-dimensional ones. In the paper [14], the iterative procedure for the simulation of wear based on the exact MDR-based solution of the three-dimensional contact has been presented. In this procedure, the contact problem is solved for the one-dimensional equivalent system, which is then transformed back to three-dimensions to calculate wear. This requires applying the direct and inverse MDR transformation in each step of simulation. The resulting procedure is orders of magnitude faster than the corresponding boundary-element programs, but still too slow to be used as an interface in larger dynamical programs. In the present paper we suggest an even simpler approximate method in which the solution of the contact problem and the calculation of wear are both carried out in the one-dimensional space.

## 2. METHOD OF DIMENSIONALITY REDUCTION: A SHORT SUMMARY

The main steps of the method of dimensionality reduction are the following. Given a rotationally symmetric three-dimensional profile $z = f(r)$, we first determine the equivalent one-dimensional profile according to the rule [15, 16]

$$g(x) = |x| \int_0^{|x|} \frac{f'(r)}{\sqrt{x^2 - r^2}} \mathrm{d}r, \tag{3}$$

as illustrated in Fig. 1. The inverse transformation is given by the integral

$$f(r) = \frac{2}{\pi} \int_0^r \frac{g(x)}{\sqrt{r^2 - x^2}} \mathrm{d}x. \tag{4}$$

The profile (3) is pressed to a given indentation depth $d$ into an elastic foundation consisting of independent springs with spacing $\Delta x$ whose normal and tangential stiffness is given by

$$k_z = E^* \Delta x, \ k_x = G^* \Delta x, \tag{5}$$

where $E^*$ is the effective elastic modulus and $G^*$ is the effective shear modulus:

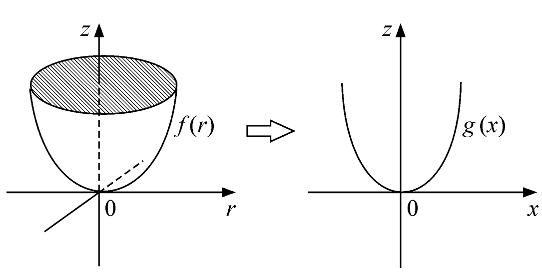


**Fig. 1.** According to the method of dimensionality reduction, the profile of a 3D body is substituted by the 1D MDR transformed profile according to a procedure described in the text.

$$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}, \ \frac{1}{G^*} = \frac{2-\nu_1}{4G_1} + \frac{2-\nu_2}{4G_2}, \tag{6}$$

$E_1$ and $E_2$ are the Young's moduli, $G_1$ and $G_2$ are the shear moduli of contacting bodies, $\nu_1$ and $\nu_2$ are their Poisson ratios. Throughout the paper, we assume that the contacting materials satisfy the condition of "elastic similarity" $(1-2\nu_1)/G_1 = (1-2\nu_2)/G_2$ which guarantees the decoupling of the normal and tangential contact problems [17]. The resulting vertical displacements of springs are given by $u_z(x) = d - g(x)$. The contact radius $a$ is given by the condition $u_z(a) = 0$ or

$$g(a) = d. \tag{7}$$

If the normal displacement of a single spring is equal to $u_z(x)$ and tangential displacement to $u_x(x)$ then the normal and tangential spring forces are equal to

$$\Delta F_z = E^* u_z(x) \Delta x \text{ and } \Delta F_x = G^* u_x(x) \Delta x \tag{8}$$

correspondingly. The total normal load $F_\mathrm{n}$ can be calculated as

$$F_\mathrm{n} = \int_{-a}^{a} E^* u_z(x) \mathrm{d}x = 2\int_0^a E^*[d - g(x)]\mathrm{d}x. \tag{9}$$

If the profile is moved tangentially by $u_x^{(0)}$, the springs will be stressed both in the normal and tangential direction, and the radius $c$ of the stick region will be given by the condition that the tangential force $\Delta F_x = k_x u_x^{(0)}$ is equal to the coefficient of friction $\mu$ multiplied with the normal force: $\Delta F_z(c) = k_z u_z(c)$ which results in the relation

$$G^* u_x^{(0)} = \mu E^* (d - g(c)). \tag{10}$$

As shown in [18], this result reproduces correctly the relations in the corresponding three-dimensional contact.

## 3. LIMITING SHAPE OF WEAR PROFILE AND DEVELOPMENT OF INTERMEDIATE SHAPES

If profile is subjected to oscillations with a small amplitude, then the inner part of the contact area with the radius $c$ given by Eq. (10) will sticking while the outer regions will slip [19–21]. In these outer regions of the contact area, wear will occur. If oscillations continue very long time, the wear profile will be tending towards a limiting shape [22]. This shape was calculated in the recent paper [23]. In particular, it was shown that the limiting form of the one-dimensional image of the method of dimensionality reduction has the form

$$g_\infty(x) = \begin{cases} g_0(x) & \text{for } 0 < x < c, \\ d & \text{for } c < x < a \end{cases} \tag{11}$$

and the correspondent shape of the three-dimensional profile has the form

$$f_\infty(r)=\begin{cases} f_0(r) \text{ for } 0<r<c, \\ \dfrac{2}{\pi}\displaystyle\int_0^c \frac{g_0(x)}{\sqrt{r^2-x^2}}\mathrm{d}x+\frac{2}{\pi}d\int_c^r\frac{1}{\sqrt{r^2-x^2}}\mathrm{d}x \\ \text{for } c<r<a. \end{cases} \quad (12)$$

The contact radius in the limiting state $\tilde{a}(c)$ is determined by the condition

$$\frac{2}{\pi}\int_0^c \frac{g_0(x)}{\sqrt{r^2-x^2}}\mathrm{d}x+\frac{2}{\pi}d\int_c^a\frac{1}{\sqrt{r^2-x^2}}\mathrm{d}x=f_0(a). \quad (13)$$

Development of the profiles between the initial and the limiting states calculated using the method proposed in [14] is illustrated with an example in Fig. 2.

## 4. APPROXIMATE RULE FOR THE WORN SHAPE OF ONE-DIMENSIONAL MDR-TRANSFORMED PROFILE

The development of the shape of one-dimensional images as shown in Fig. 2b looks simpler than that of true three-dimensional profile. It is easy to "mimic" this development if we note that the main tendency of the profile in Fig. 2b is just tending to the constant value of $d$ everywhere in the interval $c<x<a$. We can try to simulate this development by the equation

$$\frac{\mathrm{d}g(x)}{\mathrm{d}\delta u_x}=-\frac{\xi k}{a\sigma_0}E^*(g(x)-d) \text{ for } c<x<a(c), \quad (14)$$

where $\delta u_x(x)$ is the relative displacement of the bodies in contact, $a(c)$ is the solution of Eq. (13) and $\xi$ is a dimensionless fitting parameter of the order of unity. As $E^*(g(x)-d)$ is the linear force density, and $E^*(g(x)-d)/a$ has the order of magnitude of pressure, this equation can be interpreted as a one-dimensional modification of the wear law (2). However, we would like to stress that this equation should not be over interpreted as a real "wear equation", as we have to do with the formal one-dimensional image of the method of dimensionality reduction and not with the actual three-dimensional profile. For example, according to Eq. (14), the "wear rate" outside the contact radius (but inside the radius $\tilde{a}$) is non-zero, and even negative!

The procedure for the determination of the relative displacement $\delta u_x(x)$ in Eq. (14) is described in the following. Assume that the upper body oscillates periodically with a frequency $\omega$ and an amplitude $U^{(0)}$:

$$u_x^{(0)}=U^{(0)}\cos(\omega t). \quad (15)$$

As long as the tangential elastic force $\Delta F_x=k_x u_x(x)$ is smaller than the local maximum friction force $\mu\Delta F_z(x)$, the indenter sticks to the substrate; therefore, the spring displacement coincides with the displacement of the oscillating indenter. After achieving the maximum value of $\mu\Delta F_z(x)$, the tangential force does not increase further, so that the condition $\Delta k_x u_x(x)=\mu\Delta F_z(x)$ is fulfilled, and the bodies slide against each other. These conditions can be written in the form:

$$\begin{cases} \Delta u_x(x)=\Delta u_x^{(0)}, \text{ if } |f_x|=|k_x\Delta u_x(x)|<\mu f_z(x), \\ u_x(x)=\pm\dfrac{\mu f_z(x)}{\Delta k_x}, \text{ when sliding.} \end{cases} \quad (16)$$

This equation determines unambiguously the tangential displacement $u_x(x)$ of any spring and thus the incremental change $\Delta u_x(x)$ of this displacement at any time. The difference $\delta u_x(x)=\Delta u_x^{(0)}-\Delta u_x(x)$ is then the relative displacement of the indenter and substrate which

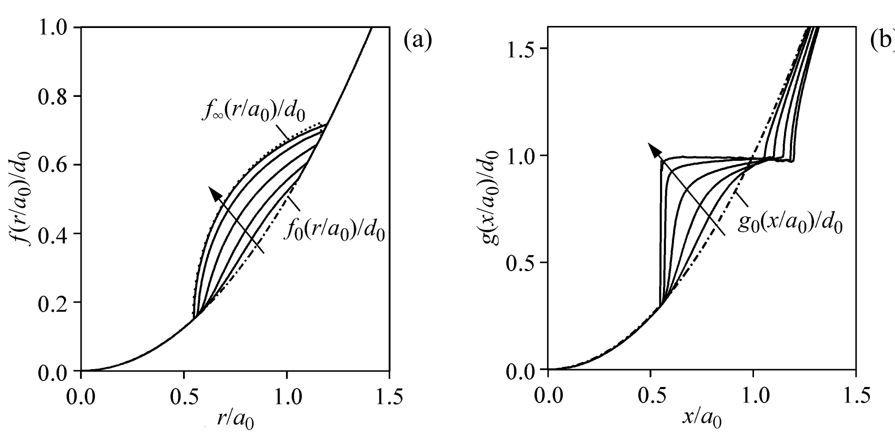


**Fig. 2.** Development of the three-dimensional profile (a) and the corresponding one-dimensional MDR image (b) due to fretting wear under conditions of constant approach of bodies (that is the indenter is pressed into the elastic half space by the indentation depth $d_0$ and then oscillates horizontally at this constant height). The amplitude of oscillations was chosen such that $c=0.55a_0$. The dimensionless number of cycles (as defined by Eq. (18)) was $\tilde{N}=4, 10, 20, 36$ and 70 as indicated by arrow.

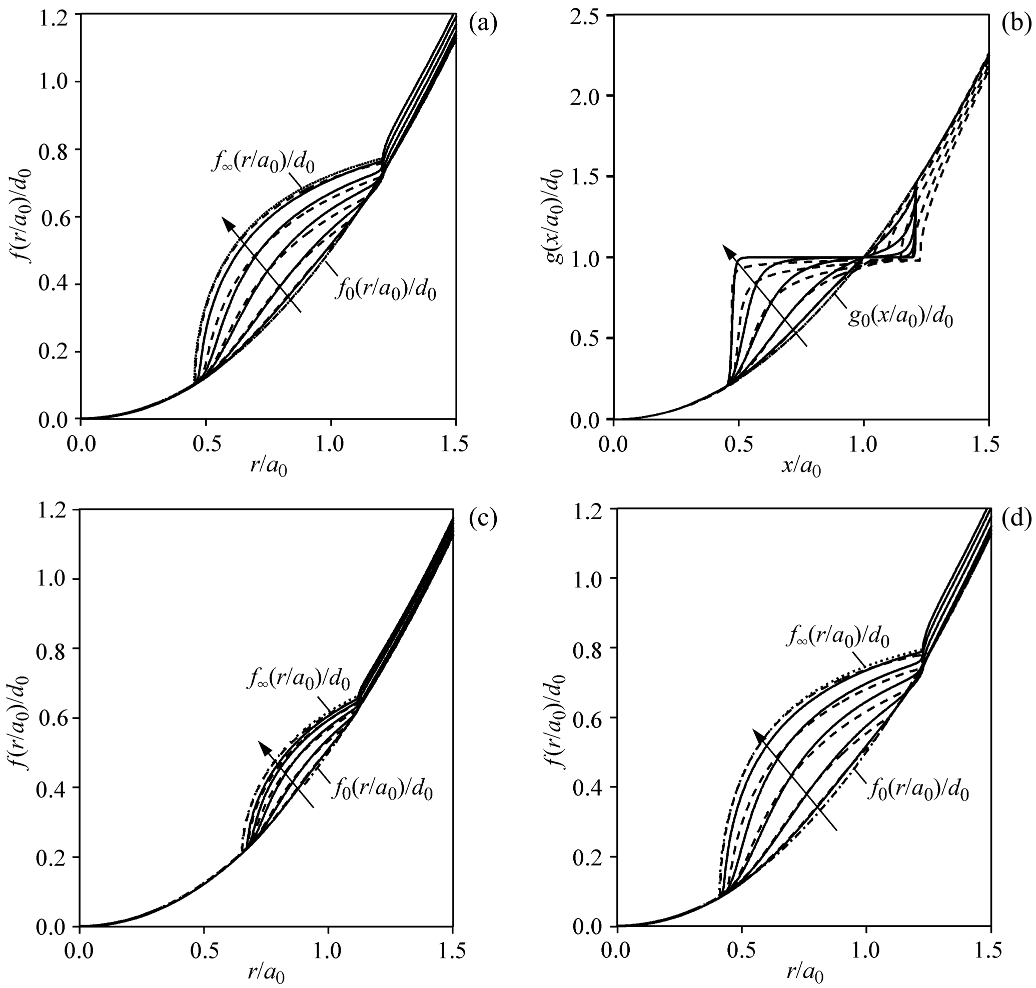


**Fig. 3.** Comparison for parabolic indenter: the three-dimensional profile obtained with Eq. (14) and subsequent inverse transformation, Eq. (4) (solid lines) (a) and the one-dimensional profile $g(x)$ calculated according to Eq. (14) (solid lines) for different number of oscillation cycles with $\xi = 0.8$; the amplitude of tangential oscillation was chosen so that $c = 0.455a_0$ (b); the three-dimensional profile obtained with Eq. (14) with smaller amplitude of tangential oscillation $U'^{(0)} = 0.8U^{(0)}$ $(c = 0.652a_0)$ (c) and larger amplitude $U'^{(0)} = 1.2U^{(0)}$ $(c = 0.41a_0)$, where $U^{(0)}$ is the amplitude for the case in Fig. 3a (d). Dashed lines are the three- and one-dimensional profiles calculated with the numerically exact procedure described in [14]. The number of oscillation cycles $\tilde{N} = 2, 8, 18, 32, 72$ as indicated by arrow, and the last line $(\tilde{N} = 72)$ in Fig. 3a, c, d almost coincides with the limiting profile from analytical solution (dot line) [23].

has to be used in the one-dimensional "wear equation" (14). Outside the contact, $\delta u_x(x) \equiv \Delta u_x^{(0)}$.

For presentation of results, we will use in this paper the following dimensionless variables. Let us denote the indentation depth of the initial profile with $d_0$ and the corresponding initial contact radius with $a_0$. All vertical coordinates will be normalized by $d_0$ and the horizontal coordinates by $a_0$. Thus, we will use the following dimensionless variables:

$$\tilde{f} = f/d_0,\ \tilde{d} = d/d_0,\ \tilde{r} = r/a_0,\quad \tilde{x} = x/a_0,\ \tilde{c} = c/a_0. \tag{17}$$

The dimensionless number of cycles is defined as

$$\tilde{N} = \frac{N}{N_0} \text{ with } N_0 = \frac{a_0 \sigma_0}{4U^{(0)} k E^*}. \tag{18}$$

For illustration of the procedure described by Eqs. (14), (16) let us consider the cases of a parabolic and a conical indenter. For the case of parabolic indenter, the initial three dimensional profile is $f_0(r) = r^2/(2R)$, where $R$ is the curvature radius. We consider the situation when this profile is indented in an elastic half space by the indentation depth $d_0$ and then oscillates at this constant height. The MDR transformed one-dimensional profile, according to Eq. (3), is given by $g_0(x) = x^2/R$. The initial contact radius is given by the condition

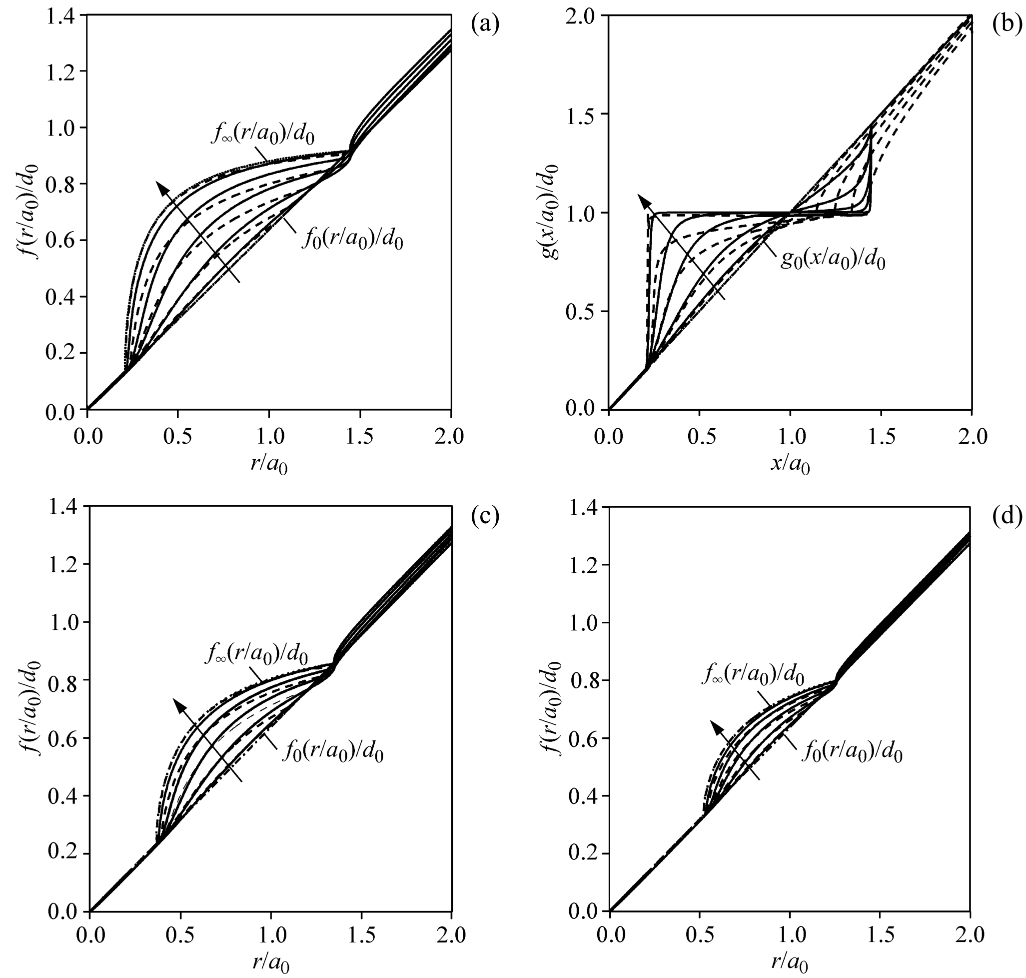


**Fig. 4.** Comparison for conical indenter: the three-dimensional profile obtained from with $g(x)$ by the inverse transformation, Eq. (4) (solid lines) (a) and the one-dimensional profile $g(x)$ calculated according to Eq. (14) (solid lines) for different number of oscillation cycles with $\xi = 0.8$, the amplitude of tangential oscillation was chosen so that $c = 0.21a_0$ (b); the three-dimensional profile obtained with Eq. (14) with the amplitude of tangential oscillation $U'^{(0)} = 0.8U^{(0)}$ $(c = 0.368a_0)$ (c) and $U'^{(0)} = 0.6U^{(0)}$ $(c = 0.522a_0)$ (d), where $U^{(0)}$ is the amplitude for the case in Fig. 4a. Dashed lines are the three- and one-dimensional profiles according to [14]. The number of oscillation cycles is $\tilde{N} = 2, 8, 18, 32, 72$ as indicated by arrow, and the last line $(\tilde{N} = 72)$ in Fig. 4a, c, d almost coincides with the limiting profile from analytical solution (dot line) [23].

$g(a_0) = d_0$. During the oscillation the stick region is determined by Eq. (10) and the contact radius is calculated as [23]

$$\tilde{a}(\tilde{c}) \approx \sqrt{\left(\frac{\tilde{c}}{2}\right)^2 + 2} - \frac{\tilde{c}}{2}. \qquad (19)$$

Now the change of the one-dimensional profile due to wear is calculated according to Eq. (14) for different number of cycles and the corresponding three-dimensional profiles are calculated by the inverse MDR-transformation (4). The resulted profiles are shown in Fig. 3a, b by solid lines. In the same figure, the results produced by the numerically exact procedure described in [14] are shown for comparison. The best fitting with exact results is achieved for $\xi = 0.8$. One can see that the approximate procedure reproduces very accurately results for the three-dimensional profile for any number of wear cycles—in any case with a better precision as the typical accuracy of wear experiments and of the used Reye–Archard–Khrushchov wear law.

For the case of conical indenter, the initial three-dimensional profile is $f_0(r) = r\tan\theta$. The corresponding MDR-transformed one-dimensional profile is $g_0(x) = \pi/2\,|x|\tan\theta$. The initial contact radius is given by the condition $g(a_0) = d_0$. During the oscillation the stick region is determined by Eq. (10) and the outer wear radius $\tilde{a}(\tilde{c})$ is calculated by solving equation [23]

$$\frac{\pi}{2} - \arcsin\left(\frac{\tilde{c}}{\tilde{a}}\right) = \sqrt{\tilde{a}^2 - \tilde{c}^2}. \qquad (20)$$

The one- and three-dimensional profiles obtained by solving Eq. (14) are shown in Fig. 4a, b by solid lines. In the same figure, the results of numerically exact procedure of paper [14] are also shown for comparison (dash lines). As for the parabolic profile, the three-dimensional shapes obtained by the present approximate procedure reproduce with good accuracy the results obtained by the numerically exact procedure of [14]. However, the calculating time is reduced by the factor of 600.

## 5. CONCLUSION

In the present paper, we suggested the simplified numerical procedure for simulation of wear of rotationally symmetric profiles, which is approximately 600 times faster than the fast MDR-based, numerically exact procedure described in [14]. Taking into account the low precision of the laws of wear, we conclude that this simplified procedure will by more than adequate for any practical simulation. Because of extreme fastness of the procedure, it can be used as a "contact and wear interface" in larger dynamic simulations.

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