Contact mechanics with the method of dimensionality reduction (MDR) I.

Lit.: V.L. Popov & M. Heß, Method of Dimensionality reduction in contact mechanics and friction: A users handbook. I. Axially-symmetric contacts, Facta Universitatis, Series: Mechanical Engineering, 2014, v. 12, N.1, pp.1-14. http://casopisi.junis.ni.ac.rs/index.php/FUMechEng/article/view/155/47 (open access)

I. List of notations:

E modulus of elasticity, G shear modulus

ν Poisson-number

$E^* = E/(1-\nu^2)$ effective modulus of elasticity

$G^* = 4G/(2-\nu)$ effective shear modulus

F or F_N normal force

a contact area, d indentation depth

$f(r)$ three-dimensional profile

$g(x)$ 1D, MDR-transformed profile

$p(r)$ pressure distribution in contact area

$q(x)$ distributed load in equivalent model

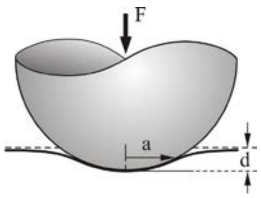
$f'(r)$, $q'(x)$ derivatives with respect to each argument

Δx distance between springs

c incremental contact stiffness

II. Contact of elastic bodies. Contacts appear in numerous applications such as clutches, brakes, tires, roller and ball bearings. Contacts can be used for transmission of mechanical forces (screws), electricity or heat or to prevent a material flow (seals).

III. Contact variables of interest. If a rigid body is pressed into an extended elastic body having a flat surface ("*elastic half-space*") with a force F, it is indented by the amount d (indentation depth) with respect to the original surface of the elastic body. Thereby a contact area with a radius a (contact radius) is formed. Inside the contact area, there is some stress distribution $p(r)$, where r is the polar radius in the contact surface. The relationships between the variables F, d, a, and the resulting stress distribution are most importantquantities which determine the contact properties.



Contact stiffness. The relationship $F(d)$ defines the contact as a "non-linear spring". If the indentation depth is changed by a small value Δd, then the normal force is changed by $\Delta F = F(d+\Delta d) - F(d)$. The ratio $c = (\Delta F/\Delta d)$ is called the *differential (or incremental) stiffness* of the contact. It determines the dynamics of the system with the point of contact (e.g. dynamics of a railway vehicle). In the contact mechanics, it is shown that the differential stiffness is determined only by the diameter $2a$ of the contact area:

$$c = 2aE^* . \tag{1}$$

Electrical conductivity. If two electrically conductive body are brought into contact with the radius a, the electrical resistance of the contact $\tilde{R}$, is equal to

$$\tilde{R} = (\rho_1 + \rho_2)/(4a), \tag{2}$$

where ρ_1 and ρ_2 describe the resistivities of the two bodies.

Heat conductivity. When two bodies with a temperature difference ΔT are brought into contact, the result is a heat flow ΔQ (unit J/s) from the warmer body to the colder one. The proportionality coefficient between the two is called the thermal resistance $R_{th} = \Delta T/\Delta Q$. Also the heat resistance is determined solely by the contact radius:

$$R_{th} = 1/\left(4a\lambda^*\right) \tag{3}$$

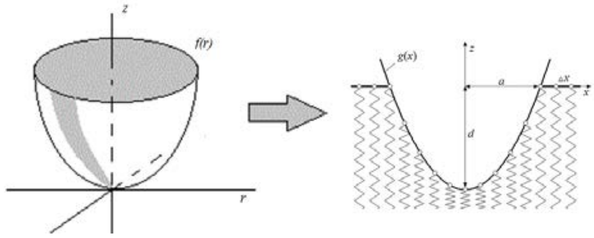
with $1/\lambda^* = 1/\lambda_1 + 1/\lambda_2$. Therein describe λ_1 and λ_2 the specific thermal conductivities of the two bodies. All three above-mentioned Variables depend on the contact radius a.

IV. Method of Dimensionality Reduction. Problem: A rigid rotationally symmetric profile $z = f(r)$ is pressed into an elastic body with the normal force F_N to a indentation depth d; the contact radius is a. We are looking for the interrelation between F_N, d and a as well as for the stress distribution in the contact.

Solution steps of the MDR:

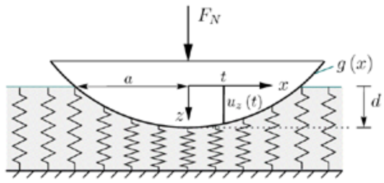
(1) We define a "MDR-transformed" one-dimensional profile

$$g(x) = |x| \int_0^{|x|} \frac{f'(r)}{\sqrt{x^2 - r^2}} \mathrm{d}r . \tag{4}$$



(2) Instead of the elastic half-space, a linear series of independent springs is defined, the so-called "elastic foundation", with a small distance Δx between the springs and normal stiffness $k_z = E^* \Delta x$ and tangential stiffness $k_x = G^* \Delta x$ of each spring.

(3) The one-dimensional profile is then pressed with the normal force F_N in the elastic foundation.



V. Calculation steps of MDR. Vertical displacement of a spring at the location x inside the contact area is equal to $u_z(x) = d - g(x)$. At the edge of the contact $x = a$ it is zero: $u_z(a) = 0 \Rightarrow$

$$d = g(a). \quad (5)$$

The corresponding spring force is equal to the displacement multiplied with stiffness:

$\Delta F_z(x) = \Delta k_z u_z(x) = E^* u_z(x) \Delta x$.

Sum of all single spring forces yields the normal force:

$$F_N := E^* \int_{-a}^{a} u_z(x) \mathrm{d}x = 2E^* \int_{0}^{a} \left(d - g(x)\right) \mathrm{d}x \quad (6)$$

We define the line load

$$q_z(x) = \frac{\Delta F_z(x)}{\Delta x} = E^* u_z(x) = E^* \left(d - g(x)\right). \quad (7)$$

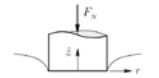
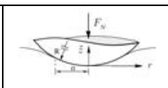
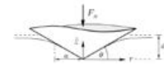
The pressure distribution in the original 3D contact is then determined according to:

$$p(r) = -\frac{1}{\pi} \int_{r}^{\infty} \frac{q_z'(x)}{\sqrt{x^2 - r^2}} \mathrm{d}x \quad (8)$$

The equations (5), (6) and (8) completely solve the contact problem.

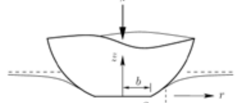
VI. Examples of profile transformation

By applying the transformation (4) you get for a flat cylindrical stamp, a sphere and a cone, the following transformed profiles:

Table I			
$f(r)$	$\begin{cases} 0, & r < a \\ \infty, & r > a \end{cases}$	$r^2 / 2R$	$r \tan\theta$
$g(x)$	$\begin{cases} 0, & x < a \\ \infty, & x > a \end{cases}$	x^2 / R	$\frac{\pi}{2} \lvert x \rvert \tan\theta$

As a less trivial example we consider the contact of *a parabolic profile with worn tip*:

$$f(r) = \begin{cases} 0 & \text{for } 0 \le r < b \\ \dfrac{r^2 - b^2}{2R} & \text{for } b \le r \le a \end{cases}$$

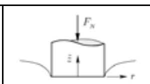
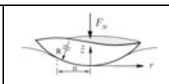
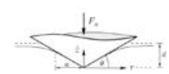


Das MDR-transformed Profile is equal to

$$g(x) = \begin{cases} 0 & \text{for } 0 \le |x| < b \\ \dfrac{|x|}{R}\sqrt{x^2 - b^2} & \text{for } b \le |x| < a \end{cases}.$$

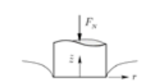
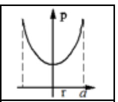
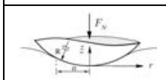
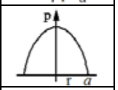
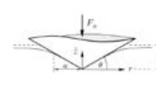
VII. Examples of the calculation of the contact radius and the normal force.

Substitution of the MDR-transformed profiles of table I in (5) gives the radius of contact a as a function of the indentation depth d. Further insertion into (6) gives the normal force as a function of the indentation depth. Results are summarized in table II.

Table II			
a	a	$\sqrt{Rd}$	$\frac{2}{\pi} \frac{d}{\tan\theta}$
F_N	$2aE^*d$	$\frac{4}{3} E^* R^{1/2} d^{3/2}$	$\frac{2}{\pi} E^* \frac{d^2}{\tan\theta}$

VIII. Examples of pressure distributions

Substitution of profiles from table I into equation (7), and then in (8) yields the following pressure distributions:

	$p(r) = \frac{E^* d}{\pi a}\left(1 - \left(\frac{r}{a}\right)^2\right)^{-1/2}$	
	$p(r) = \frac{2E^*}{\pi}\left(\frac{d}{R}\right)^{1/2} \sqrt{1 - \left(\frac{r}{a}\right)^2}$	
	$p(r) = \frac{E^*}{2} \tan\theta \cdot \ln\left(\frac{a}{r} + \sqrt{\left(\frac{a}{r}\right)^2 - 1}\right)$