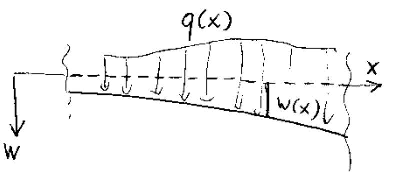
I. Beam differential equation of 4th order



The form of the neutral axis of a bent beam ("elastic line") is calculated either using equation

$$EI_z w''(x) = -M_z(x) \qquad (1)$$

(provided the bending moment can be determined as a function of the coordinate in advance, i.e. for statically determinate systems) or from the *beam differential equation of 4th order*.

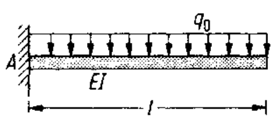
$$\boxed{\left(EI_z w''(x)\right)'' = q(x)\,. \qquad (2)}$$

This equation can also be applied to statically indeterminate systems. The four integration constants must be determined from four boundary conditions.

II. Examples

We examine three identical beams of constant flexural rigidity *EI* under constant line load q_0 but differently supported at the ends.

B1. This cantilever beam is statically determinate. You could first calculate the torque curve and then apply equation (1). Even easier is easier to use the equation (2):



$$EIw^{IV} = q_0\,.$$

Four integrations yield:

$$EIw''' = -Q = q_0 x + C_1\,,$$

$$EIw'' = -M = \tfrac{1}{2} q_0 x^2 + C_1 x + C_2\,,$$

$$EIw' = \tfrac{1}{6} q_0 x^3 + \tfrac{1}{2} C_1 x^2 + C_2 x + C_3\,,$$

$$EIw = \tfrac{1}{24} q_0 x^4 + \tfrac{1}{6} C_1 x^3 + \tfrac{1}{2} C_2 x^2 + C_3 x + C_4\,.$$

The boundary conditions are: $w(0) = 0$, $w'(0) = 0$, $M(l) = -EIw''(l) = 0$, $Q(l) = -EIw'''(l) = 0$.

From the first two it follows $C_3 = 0$, $C_4 = 0$.

The last two constraints are

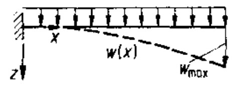
$$q_0 l + C_1 = 0\,,$$

$$\tfrac{1}{2} q_0 l^2 + C_1 l + C_2 = 0\,.$$

Hence: $C_1 = -q_0 l$, $C_2 = \tfrac{1}{2} q_0 l^2$.

The elastic line is thus:

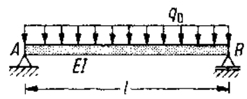
$$w(x) = \frac{q_0 x^2}{EI}\left(\tfrac{1}{24} x^2 - \tfrac{1}{6} l x + \tfrac{1}{4} l^2\right).$$



The maximum lowering is achieved at the end point and is equal to

$$w(l) = \frac{q_0 l^2}{EI}\left(\tfrac{1}{24} l^2 - \tfrac{1}{6} l^2 + \tfrac{1}{4} l^2\right) = \tfrac{1}{8}\frac{q_0 l^4}{EI}\,.$$

B2. A simply supported beam is also statically determinate. The Eq. (1) *can* be used. However, it is easier to use the equation (2). Since the line load is the same as in Example 1, the general solution is the same too. The only difference lies in the boundary conditions:



$w(0) = 0$, $w(l) = 0$, $M(0) = 0 \;\Rightarrow\; w''(0) = 0$,

$M(l) = 0 \;\Rightarrow\; w''(l) = 0$.

$$EIw(0) = C_4 = 0\,, \qquad EIw''(0) = C_2 = 0\,,$$

$$EIw(l) = \tfrac{1}{24} q_0 l^4 + \tfrac{1}{6} C_1 l^3 + C_3 l = 0\,,$$

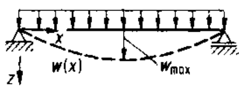
$$EIw''(l) = \tfrac{1}{2} q_0 l^2 + C_1 l = 0\,,$$

$$C_1 = -\tfrac{1}{2} q_0 l\,,\; C_3 = \tfrac{1}{24} q_0 l^3\,.$$

The elastic line is

$$w(x) = \tfrac{1}{EI}\left(\tfrac{1}{24} q_0 x^4 - \tfrac{1}{12} q_0 l x^3 + \tfrac{1}{24} q_0 l^3 x\right)$$

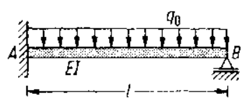
$$= \tfrac{q_0}{24EI}\left(x^4 - 2lx^3 + l^3 x\right)$$



The maximum deflection is achieved in the center, $x = l/2$, and is equal to:

$$w(l/2) = \frac{q_0}{24EI}\left(\tfrac{1}{16} l^4 - \tfrac{1}{4} l^4 + \tfrac{1}{2} l^4\right) = \frac{5}{384}\frac{q_0 l^4}{EI}$$

B3. The beam shown below is mounted *statically indeterminate*. Sectional loads (including the bending moment) can therefore not be determined solely from the equilibrium conditions. The use of equation (2) is the only possible way of calculating the elastic line in this case. We assume that the beam was mounted stress free in the absence of line load.



Since the line load is the same as in Example 1, the general solution is the same. The only difference, again, are the the boundary conditions:

$w(0) = 0$, $w'(0) = 0$, $w(l) = 0$,

$M(l) = -EIw''(l) = 0$.

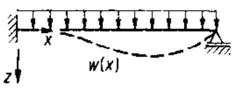
$$EIw(0) = C_4 = 0\,, \quad EIw'(0) = C_3 = 0\,,$$

$$EIw(l) = \tfrac{l^2}{2}\left(\tfrac{1}{12} q_0 l^2 + \tfrac{1}{3} C_1 l + C_2\right) = 0\,,$$

$$EIw''(l) = \tfrac{1}{2} q_0 l^2 + C_1 l + C_2 = 0\,.$$

From the last two equation follows:

$C_1 = -\frac{5}{8} q_0 l$, $C_2 = \frac{1}{8} q_0 l^2$.



For the bending line, this results in:

$$w(x) = \frac{q_0}{EI}\left(\frac{1}{24} x^4 - \frac{5}{48} l x^3 + \frac{1}{16} l^2 x^2\right).$$

Together with the bending line we have also determined the dependencies of $w'(x)$, of the bending moment $M(x) = -\frac{1}{8} q_0 \left(4x^2 - 5lx + l^2\right)$ and of the shear force $Q(x) = -\frac{1}{8} q_0 \left(8x - 5l\right)$.

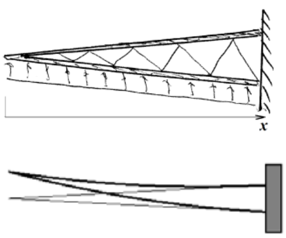
For the support reactions, we get:

$A = Q(0) = \frac{5}{8} q_0 l$, $B = -Q(l) = \frac{3}{8} q_0 l$,

$M^{(A)} = M(0) = -\frac{1}{8} q_0 l^2$.

Elastic line of a beam with variable bending stiffness

B1. A wing has the structure shown below and is loaded with a constant line load. The area



moment of inertia of the cross section can be written as $I(x) = I_0 x^2 / l^2$. The bending differential equation has theform: $\left(EIw''(x)\right)'' = -q_0$. Its double integration, taking into account the boundary conditions $\left(EIw''(x)\right)'\Big|_{x=0} = 0$ and $EIw''(x)\Big|_{x=0} = 0$ results in

$$EI_0 \left(x/l\right)^2 w''(x) = -\frac{1}{2} q_0 x^2$$

or $EI_0 w''(x) = -\frac{1}{2} q_0 l^2$.

Another two integrations yield:

$EI_0 w'(x) = -\frac{1}{2} q_0 l^2 x + C_3$,

$EI_0 w(x) = -\frac{1}{4} q_0 l^2 x^2 + C_3 x + C_4$.

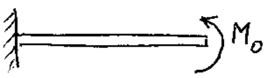
From the boundary conditions $w(l) = 0$ and $w'(l) = 0$, it follows

$C_3 = \frac{1}{2} q_0 l^3$, $C_4 = -\frac{1}{4} q_0 l^4$.

The bend line is therefore a parabola.

$$w(x) = -\frac{q_0 l^2}{4EI_0}\left(x - l\right)^2.$$

B5. Elastic line under the action of a moment:



The general solution is: $EIw''' = -Q = C_1$,

$EIw'' = -M = C_1 x + C_2$,

$EIw' = \frac{1}{2} C_1 x^2 + C_2 x + C_3$,

$EIw = \frac{1}{6} C_1 x^3 + \frac{1}{2} C_2 x^2 + C_3 x + C_4$.

The boundary conditions are: $w(0) = 0$,

$w'(0) = 0$, $M(l) = -EIw''(l) = M_0$;

$Q(l) = -EIw'''(l) = 0$. Daraus folgt:

$EIw(0) = C_4 = 0$, $\quad EIw'(0) = C_3 = 0$,

$EIw''(l) = C_1 l + C_2 = -M_0$,

$EIw'''(l) = -Q = C_1 = 0$.

Bending line is a parabola: $w(x) = -\dfrac{1}{2}\dfrac{M_0}{EI} x^2$.

Example 6. At a left firmly clamped beam acts a linearly increasing line load. Determine the bending line. <u>Solution:</u> The beam differential equation of the fourth order is:

$EIw^{IV} = q_0 x / l$.

Its four-fold integration yields:

$EIw''' = -Q = \frac{1}{2} q_0 x^2 / l + C_1$,

$EIw'' = -M = \frac{1}{6} q_0 x^3 / l + C_1 x + C_2$,

$EIw' = \frac{1}{24} q_0 x^4 / l + \frac{1}{2} C_1 x^2 + C_2 x + C_3$,

$EIw = \frac{1}{120} q_0 x^5 / l + \frac{1}{6} C_1 x^3 + \frac{1}{2} C_2 x^2 + C_3 x + C_4$.

The boundary conditions are: $w(0) = 0$,

$w'(0) = 0$, $M(l) = -EIw''(l) = 0$,

$Q(l) = -EIw'''(0) = 0$.

Of the first two follows $C_3 = 0$ and $C_4 = 0$.

The last two boundary conditions are:

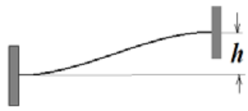
$EIw'''(l) = \frac{1}{2} q_0 l + C_1 = 0$,

$EIw''(l) = \frac{1}{6} q_0 l^2 + C_1 l + C_2 = 0$.

Hence: $C_1 = -\frac{1}{2} q_0 l$ and $C_2 = \frac{1}{3} q_0 l^2$.

$$\boxed{w(x) = \frac{1}{EI}\left(\frac{1}{120} q_0 x^5 / l - \frac{1}{12} q_0 l x^3 + \frac{1}{6} q_0 l^2 x^2\right)}.$$

B7.



The general solution:

$w' = \frac{1}{2} C_1 x^2 + C_2 x + C_3$

$w = \frac{1}{6} C_1 x^3 + \frac{1}{2} C_2 x^2 + C_3 x + C_4$.

Boundary conditions: $w(0) = 0$, $w'(0) = 0$,

$w(l) = -h$, $w'(l) = 0$.

$$w(x) = h\left(2\left(x/l\right)^3 - 3\left(x/l\right)^2\right).$$