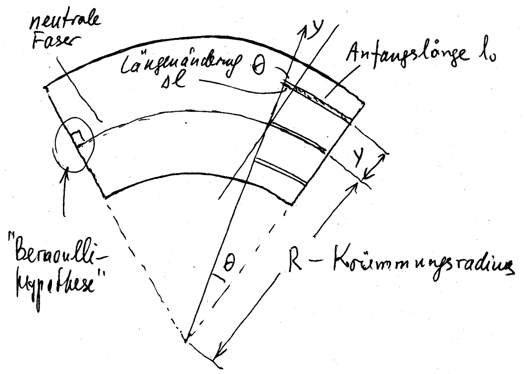
Mechanics I / Dr. J. Starcevic/Prof. Popov / Lecture 17. **Mechanics of materials**

Bending of beams.

Literature: ***Hauger, Schnell und Groß. Engineering Mechanics*** **2 (Elastostatics), 4.1., 4.3**

I. Bending moment in a bent beam. In a bent bar, the material is compressed on the inner side of the curvature and is stretched on the outer side. In between, there must be a "neutral surface" where the material is not stretched.

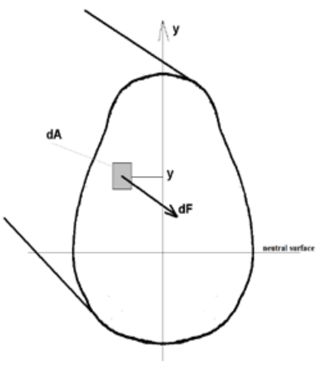


Under pure bending (under the action of a bending moment) the cross-sections of the beam remain approximately perpendicular to the beam axis (neutral surface). In bending under the action of a shear force, this condition is not exact any more, but is a good approximation for slender beams ("Bernoulli hypothesis").

Consider an infinitesimally small element of the beam with the angle measure θ (as seen from the center of curvature). We divide this element into thin strips parallel to the beam axis. For a strip with the coordinate y (measured from the neutral surface) the following relationships can be found (see the above sketch):

Initial length: $l_0 = R\theta$,

Change in length: $\Delta l = y\theta$.



For the strain we get: $\varepsilon = \dfrac{\Delta l}{l_0} = \dfrac{y}{R}$

The tensile / compressive stress is given by the Hooke's law

$$\sigma = E\varepsilon = E\frac{y}{R}.$$

The total force acting in the cross-section is:

$$N = \int_{Cross-section} dF = \frac{E}{R}\int y\,\mathrm{d}A. \qquad (1)$$

The moment of an infinitesimal force dF is equal to

$$dM_z = -dF \cdot \underbrace{y}_{Leverarm} = -\frac{E}{R}y^2 dA.$$

The total moment is equal to

$$M_z = \int dM_z = -\frac{E}{R}\int y^2 dA. \qquad (2)$$

But: Where is the neutral surface? (So far, we have chosen an arbitrary location). The position of the neutral surface is determined by (1). If the longitudinal force is zero: $N = 0$, then it follows $\int y dA = 0$. This means that the neutral surface passes *through the centroid of the cross section*.

[This follows from the definition of the centroid coordinate $y_s = \dfrac{\int y dA}{\int dA}$].

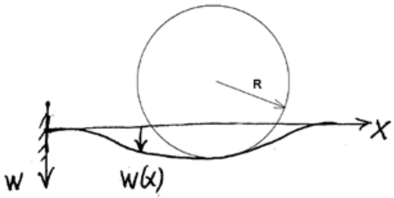
The quantity $\boxed{I_z = \int y^2 dA}$ is called geometrical moment of inertia of the cross section relative to the z-axis. With this notation, Eq. (2) for the bending moment takes the form:

$$\boxed{M_z = -\frac{EI_z}{R}}$$

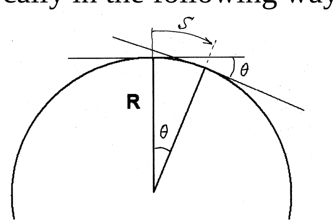
This is the fundamental eq. of the beam theory.

II. Elastic bending line.

A. A little bit geometry:



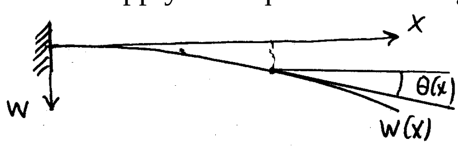
At any point of a curved bar one can define the local radius of curvature. The radius of curvature can be calculated analytically in the following way. Let us examine the rotation of the tangent to a circle having radius R:



$$\Delta\theta = \frac{\Delta s}{R}.$$

This implies that $\dfrac{1}{R} = \dfrac{\Delta\theta}{\Delta s}$. For non-constant curvature thus should be replaced through the differential quotient $\dfrac{1}{R} = \dfrac{d\theta(s)}{ds}$.

Let us apply this eq. to the bending line $w(x)$:



$$\theta(x) \approx \tan\theta(x) = \frac{dw(x)}{dx} = w'(x),$$

$s \approx x \Rightarrow$

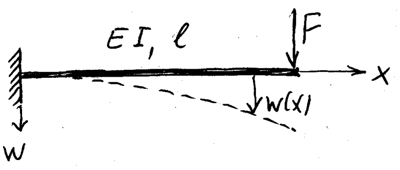
$$\frac{1}{R} = \frac{d\theta}{ds} \approx \frac{d\theta}{dx} \approx \frac{d}{dx}\left(\frac{dw}{dx}\right) = \frac{d^2w}{dx^2} = w''(x).$$

For the bending moment, this results in

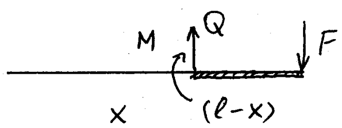
$$M_z = -\frac{EI_z}{R} = -EI_z w''(x)$$

"Beam equation"

III. Classic example: bending of a cantilever beam under an applied force at its end.



First, we make a free body sketch and determine the dependency of cut loads:



$Q = F$,

$M = -F(l-x)$.

From the beam equation, it follows

$$\frac{d^2w}{dx^2} = -\frac{M}{EI} = \frac{F(l-x)}{EI}.$$

The first integration yields:

$$\frac{dw}{dx} = \int \frac{F(l-x)}{EI}dx + C_1 = \frac{F}{EI}\left(lx - \frac{x^2}{2}\right) + C_1.$$

The second integration yields:

$$w = \int \left[\frac{F}{EI}\left(lx - \frac{x^2}{2}\right) + C_1\right]dx + C_2 =$$

$$= \frac{Fl}{EI}\frac{x^2}{2} - \frac{F}{EI}\frac{x^3}{6} + C_1 x + C_2$$

Boundary conditions:

Tabelle 4.2. Randbedingungen

Lager	w	w'	M	Q
gelenkiges Lager	0	$\neq 0$	0	$\neq 0$
Parallelführung	$\neq 0$	0	$\neq 0$	0
Einspannung	0	0	$\neq 0$	$\neq 0$
freies Ende	$\neq 0$	$\neq 0$	0	0

For a beam which is firmly clamped at the left end applies:

$w(0) = 0$, $w'(0) = 0$.

Hence: $C_1 = 0$, $C_2 = 0$.

The bending line has the form:

$$w(x) = \frac{Fl}{EI}\frac{x^2}{2} - \frac{F}{EI}\frac{x^3}{6}$$

The lowering of the point of application of force is equal to:

$$w(l) = \frac{Fl^3}{3EI}.$$

Example: Spring stiffness of a leaf spring:

$$c = \frac{F}{w(l)} = \frac{3EI}{l^3}.$$

IV. Beam under a distributed load.

In the case of continuously distributed transverse force, the following relation exist between the bending moment and the line force density $q(x)$:

$$\frac{d^2M(x)}{dx^2} = -q(x).$$

We take the beam equations:

$M(x) = -EI_z w''(x)$

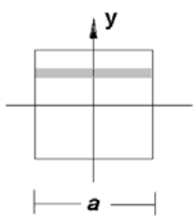
and differentiate it twice with respect to the coordinate x :

$$\left(EI_z w''(x)\right)'' = q(x)$$

(Bar differential equation of 4th order)

For a homogeneous beam it simplifies to:

$$EI_z w^{IV}(x) = q(x).$$



V. Area moment of inertia of a beam with a square cross-section.

$$I_z = \int y^2 dA = \int_{-a/2}^{a/2} y^2 a\,dy$$

$$= a\left.\frac{y^3}{3}\right|_{-a/2}^{a/2} = 2a\frac{a^3}{3\cdot 8} = \frac{a^4}{12}$$

Spring stiffness of a "spring bar" with a square cross-section:

$$c = \frac{3EI}{l^3} = \frac{Ea^4}{4l^3}$$