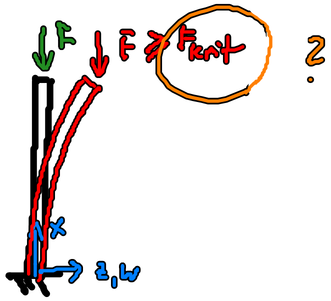
15. Übung – Knicken



DGL für Knicken: $\left(EI\, w''(x)\right)'' = -F\, w''(x)$

$$EI\, w^{IV}(x) + F\, w''(x) = 0 \quad \Big| : EI$$

$$w^{IV}(x) + \underbrace{\frac{F}{EI}}_{\lambda^2} w''(x) = 0$$

$$\boxed{w^{IV}(x) + \lambda^2 w''(x) = 0} \quad (1)$$

allgemeine Lösung:

$$w(x) = A\cos(\lambda x) + B\sin(\lambda x) + \underset{(Cx)}{C\lambda x} + D \quad (2)$$

$$w'(x) = -\lambda A\sin(\lambda x) + \lambda B\cos(\lambda x) + C\lambda$$

$$w''(x) = -\lambda^2 A\cos(\lambda x) - \lambda^2 B\sin(\lambda x)$$

$$w'''(x) = \lambda^3 A\sin(\lambda x) - \lambda^3 B\cos(\lambda x)$$

$$w^{IV}(x) = \lambda^4 A\cos(\lambda x) + \lambda^4 B\sin(\lambda x) = \lambda^2\left(\lambda^2 A\cos(\lambda x) + \lambda^2 B\sin(\lambda x)\right)$$

$$w^{IV}(x) = -\lambda^2 w''(x) \quad \Big| + \lambda^2 w''(x)$$

$$w^{IV}(x) + \lambda^2 w''(x) = 0$$

Vorgehen:

- 4 RBen bestimmen, FS in ausgelenkter Lage!
- RBen einsetzen
- Matrix aufstellen $\underline{K}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \vec{0}$
- $\det(\underline{K}) \overset{!}{=} 0 \Rightarrow$ Eigenwertgleichung
- kleinstes $\lambda \neq 0$ ablesen

$\Rightarrow F_{krit} = EI\,\lambda_1^2$

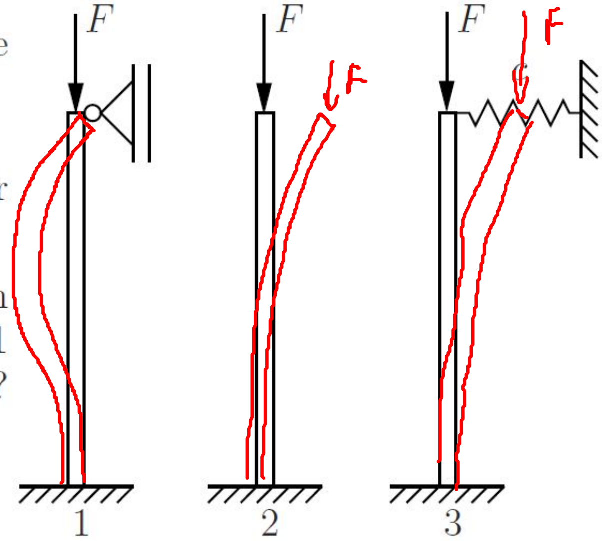
$\lambda^2 = \frac{F}{EI}$

148. Ein Balken der Biegesteifigkeit EI und der Länge l werde auf verschiedene Arten gelagert.

(a) Berechnen Sie die kritischen Lasten F_{krit} für die Varianten 1 bis 3!

(b) Wie müsste das Verhältnis der Balkenlängen l_1/l_2 gewählt werden, damit für Variante 1 und 2 dieselbe kritische Last berechnet wird?

Geg.: F, c, l, EI



a) Man erkennt:

- Fall 1 entspricht Fall 3 mit $c \to \infty$
- Fall 2 entspricht Fall 3 mit $c = 0$

$\Rightarrow$ Fall 3 lösen, $c=0$, $c \to \infty$ einsetzen!

1) DGL:

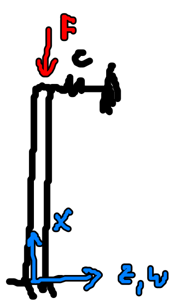
$$(EI\,w'')'' + F\,w'' = 0$$

$$EI\,w'''' + F\,w'' = 0 \quad | :EI$$

$$w'''' + \lambda^2 w'' = 0, \quad \text{mit } \lambda^2 = \frac{F}{EI}$$

2) allg. Lösung: $w(x) = A\cos(\lambda x) + B\sin(\lambda x) + C\lambda x + D$

3) RBen:

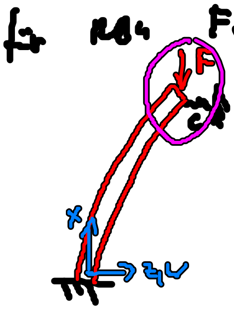


$w(x=0) = 0$ RB1

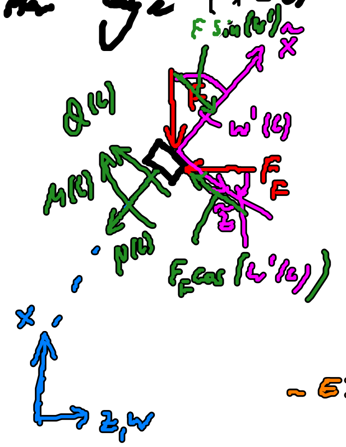
$w'(x=0) = 0$ RB2

$M(x=l) = 0$ RB3

für RB4 FS in ausgelenkter Lage $(x=l)$:



FS:



GGB:

$$\tilde{z}: 0 = -Q(l) + F\sin(w'(l)) - F_F\cos(w'(l))$$

$$\underbrace{Q(l)}_{\sim -EI w'''(l)} = F\sin(\underbrace{w'(l)}_{w'(l)}) - F_F\cos(\underbrace{w'(l)}_{\approx 1})$$

mit:

$$Q(l) = -EI w'''(l)$$

$$\underset{\alpha \ll 1}{\sin(\alpha) \approx \alpha}, \quad \cos(\alpha) \approx 1$$

$$F_F = c \cdot \Delta s = c \cdot w(l), \quad \Delta s = w(l)$$

$$\Rightarrow -EI w'''(l) = F w'(l) - c w(l) \quad \text{RB4}$$

4) Einsetzen:

$$\boxed{\begin{aligned}
w(x) &= A\cos(\lambda x) + B\sin(\lambda x) + C\lambda x + D \\
w'(x) &= -\lambda A\sin(\lambda x) + \lambda B\cos(\lambda x) + C\lambda \\
w''(x) &= -\lambda^2 A\cos(\lambda x) - \lambda^2 B\sin(\lambda x) \\
w'''(x) &= \lambda^3 A\sin(\lambda x) - \lambda^3 B\cos(\lambda x) \\
w''''(x) &= \lambda^4 A\cos(\lambda x) + \lambda^4 B\sin(\lambda x)
\end{aligned}}$$

RB1: $w(0) = 0 = A + 0 + 0 + D$

$$D = -A \quad (1)$$

RB2: $w'(0) = 0 = \lambda B + C\lambda$

$$C\lambda = -\lambda B \quad (2)$$

RB3: $M(l) = 0 \Longrightarrow -\cancel{EI} w''(l) = 0 = -\lambda^2 A\cos(\lambda l) - \lambda^2 B\sin(\lambda l) \quad (3)$

RB4: $EI w'''(l) + F w'(l) - c w(l) = 0$

$$\cancel{EI}(\lambda^3 A\sin(\lambda l) - \lambda^3 B\cos(\lambda l)) + \underbrace{\frac{F}{EI}}_{\lambda^2}(-\lambda A\sin(\lambda l) + \lambda B\cos(\lambda l) + \underbrace{\cancel{C\lambda}}_{-\lambda B\ (2)})$$

$$-\frac{c}{EI}(A\cos(\lambda l) + B\sin(\lambda l) - \lambda B l + \underbrace{\cancel{D}}_{-A\ (1)}) = 0 \quad (4) \quad \Big| : EI$$

$$\lambda^3(\cancel{A\sin(\lambda l)} - \cancel{B\cos(\lambda l)}) + \lambda^3(\cancel{-A\sin(\lambda l)} + \cancel{B\cos(\lambda l)} - B)$$

$$-\frac{c}{EI}(A\cos(\lambda l) + B\sin(\lambda l) - \lambda B l - A) = 0$$

$$-\lambda^2 B - \frac{c}{EI}\Big(\underbrace{\cancel{A\cos(\lambda l) + B\sin(\lambda l)}}_{(3)'} - \lambda B l - A\Big) = 0 \quad (4)'$$

$$-\cancel{\lambda^2} A\cos(\lambda l) - \cancel{\lambda^2} B\sin(\lambda l) = 0 \quad \Big|:\lambda^2$$

$$+A\cos(\lambda l) + B\sin(\lambda l) = 0 \quad (3)'$$

$$-\lambda^3 B - \frac{c}{EI}(-\lambda B l - A) = 0$$

$$\frac{c}{EI} A + \left(\frac{c}{EI}\lambda l - \lambda^3\right) B = 0 \quad (4)'$$

5) Matrix:

$(3)', (4)' \Longrightarrow$

$$\underbrace{\begin{bmatrix} \cos(\lambda l) & \sin(\lambda l) \\ \frac{c}{EI} & \frac{c}{EI}\lambda l - \lambda^3 \end{bmatrix}}_{\underline{K}} \begin{pmatrix} A \\ B \end{pmatrix} = \vec{0}$$

Es gibt nur dann nicht-triviale Lösung wenn $\det(\underline{K}) = 0$

$$\det(\underline{K}) = \cos(\lambda l)\left(\frac{c}{EI}\lambda l - \lambda^3\right) - \frac{c}{EI}\sin(\lambda l) = 0 \quad \Big| + \frac{c}{EI}\sin(\lambda l)$$

$$\cos(\lambda l)\left(\frac{c}{EI}\lambda l - \lambda^3\right) = \frac{c}{EI}\sin(\lambda l) \quad \Big| \cdot \frac{EI}{c}\ ;\ :\cos(\lambda l)$$

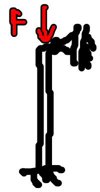
$$\lambda l - \lambda^3 \frac{EI}{c} = \tan(\lambda l)$$

$$\boxed{\lambda l\left(1 - (\lambda l)^2 \frac{EI}{l^2 c}\right) = \tan(\lambda l)} \quad (5)$$

Eigenwertgleichung!

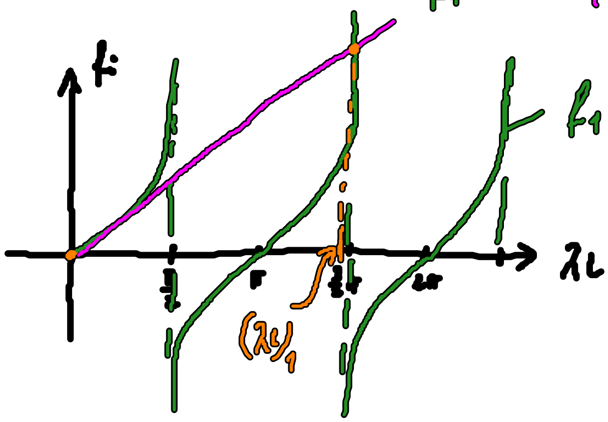
6.) Fall bestimmen

Fall 1:



$c \to \infty$ in (5):

$$\underbrace{\tan(\lambda l)}_{f_1} = \underbrace{\lambda L}_{f_2}$$



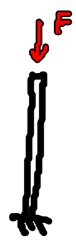
Ablesen: $(\lambda l)_1 \approx 4{,}5 \Rightarrow \lambda_1 = \frac{4{,}5}{L}$

kritischer Eigenwert $\lambda l \neq 0$

damit: $F_{krit} = EI(\lambda_1)^2$

$$F^1_{krit} = \frac{EI}{L^2}(4{,}5)^2$$

Fall 2:

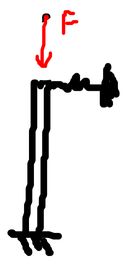


$c = 0$ in (5)

$$\tan(\lambda l) = \lim_{c \to 0} \left(\lambda l\left(1 - (\lambda L)^2 \frac{EI}{cL^3}\right)\right) = -\infty$$

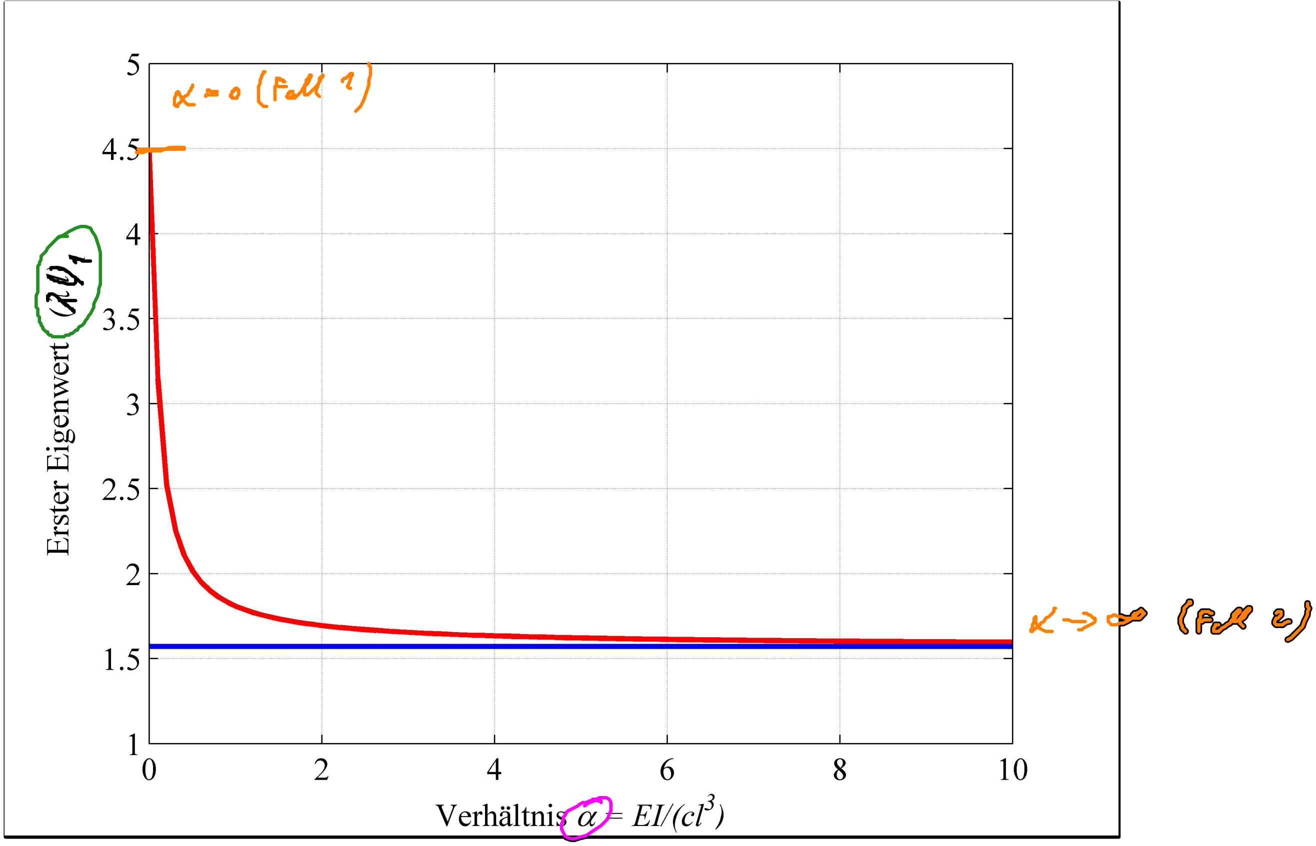
damit: $(\lambda l)_1 = \frac{\pi}{2} \Rightarrow F^2_{krit} = \frac{EI}{L^2}\left(\frac{\pi}{2}\right)^2$

Fall 3:



$$\tan(\lambda l) = \lambda l\left(1 - (\lambda l)^2 \underbrace{\frac{EI}{cl^3}}_{\alpha}\right)$$

$$\tan(\lambda l) = \lambda l\left(1 - \alpha(\lambda l)^2\right) \Rightarrow \text{Numerisch lösen}$$



Tut: 155, 153; **Ha:** 148, 154

Experiment:

Euler-Fall I:

$$F_{krit} = \frac{EI}{4}\,\frac{\pi^2}{l^2} \approx \frac{1}{4}\,210\cdot 10^{9}\,\frac{N}{m^2}\cdot\frac{1}{12}\cdot 10\,mm\cdot(1\,mm)^3\cdot 10^{-12}\,\frac{m^4}{mm^4}\cdot\pi^2\,\frac{1}{(1\,m)^2}$$

$$\approx 0{,}4\,N \Rightarrow m_{krit} \approx 40\,g$$