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COMMENT

Comment on "Contact Mechanics for Randomly Rough Surfaces: On the Validity of the Method of Reduction of Dimensionality" by Bo Persson in Tribology Letters

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Abstract In a recent paper, Persson (Tribol Lett, 2015. doi:10.1007/s11249-015-0498-1) has criticized the validity of the method of dimensionality reduction in contact mechanics. Attempts to determine the scope and validity of particular theories clearly are an important and necessary part of science. However, we believe that much of the criticism by Persson is misleading in both scope and particulars, and take this opportunity to present the other side of the argument. We hope that our comment can help the reader to avoid misunderstandings and to arrive at a more balanced view of the method in question.

Keywords Method of dimensionality reduction · Contact mechanics

1 Introduction

Let us start with a short notice to the reference list of this comment. As we have to analyse in detail the arguments put forward by Persson in his paper, we reproduce the complete list of references from his paper (Refs. [1–31]). Additional references of this comment start with the number [32].

In the paper [32], Persson criticizes MDR and claims that "It has recently been argued, based on comparison to exact numerical calculations, that this model fails even qualitatively to describe the contact mechanics correctly". The main logic of the Persson's paper is the following: "Complete contact" (full area of contact) in a real three-dimensional contact of rough surfaces is achieved at much larger normal force than in the MDR-based model for rough contact. "Therefore", MDR is incorrect—in all its possible applications with exception of normal contact of axis-symmetric bodies with compact contact area.

We also note that the paper [32] basically attacks our earlier claim in [7] that the contact area is correctly described by the MDR using the mapping rule by Geike. This claim really went too far and has already been explicitly withdrawn in all our publications since 2012, see, e.g. [9] and later. In the review paper [33], there was even a special section devoted to this correction and description of the region of applicability of the MDR. This correction can be found in the monograph [1] on the MDR (Introduction and all further relevant chapters). Thus, Persson's "simple argument to show that this theory fails qualitatively" is more like a mathematical trick, for it is about condition of achieving the complete "area contact", which has been excluded from consideration in the MDR model.

In passing, Persson cites as being wrong many other papers which have nothing to do with the MDR or operate in the region of its indisputable applicability and cites for his support papers which do not say even one critical word about MDR. Through this massively, let us say, *inaccurate* citation, the many-year work of many people is defamed. Even though I understand that the detailed discussion of the paper [32] is probably of no particular interest for scientific community, I am urged to analyse its statements in detail. The main aim of this comment is therefore to clarify once more the applicability region of the MDR (outside it, the validity of the MDR may fail). I start with precisely describing the areas of applicability of the MDR, and the next section will be used as a reference in the subsequent analysis.

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2 Region of Exact Validity of MDR

At the beginning of his paper, Persson states that the MDR provides exact solutions for contacts of axially symmetric bodies with compact contact area and that in the case of normal contact of elastic bodies this follows from Sneddon's solution [2]. This is correct, but it should also be noted that the MDR gives exact from the mathematical point of view results for other practically important classes of contact problems as well, including any problem that can be *reduced to the axis-symmetric normal Hertz-type contact problem of elastic bodies*. Besides the above-mentioned Class I of Hertz-type frictionless normal contact, this includes:

2.1 Class II

Tangential contact of axis-symmetric bodies with constant coefficient of friction. This extension is based on the works of Cattaneo [34], Mindlin [35], Jaeger [36], and Ciavarella [37]. The complete proof for monotonous loadings is given in [1]. For the non-monotonous loadings, we have a case study [1], where Mindlin's results on energy dissipation are reproduced. In the most general case, the proof follows from the equivalence of the MDR to the method of memory diagrams [52].

2.2 Class III

Contact of elastic bodies with adhesion in the JKR (Johnson, Kendall, and Roberts) approximation. This extension is based on the works of Borodich and Galanov [38] and Yao and Gao [39] for arbitrary axis-symmetric adhesive contact. Complete proof is given in [1] and [51].

2.3 Class IV

Normal frictionless contact of viscoelastic bodies. This extension is based on the theorems of Lee [40] and Radok [41], which reduce the contact problem for arbitrary linear viscoelastic materials to the corresponding elastic contact problem. In the paper [44], the complete proof of the validity of the MDR is given for both loading and unloading phases of indentation based on the Ting's algorithm [53].

In the paper [32], Persson then states that contact between randomly rough surfaces is neither compact nor axisymmetric and that there is therefore no reason why the 1D mapping procedure[1] should be valid in that case. In our opinion, this statement is overly broad without additional qualification. As was shown in some of the papers cited in the paper [32], the MDR-based model does describe *some* properties of rough contacts *very well*, from a practical point of view, but is inapplicable to others. In particular, it was shown that the contact stiffness of rough contacts is described with a high accuracy by the 1D model due to its construction. This should not be very surprising, as it can be seen from the following simple argument: *Any* contact between two solids modelled as elastic half-spaces (Hertz's assumption), independently of the particular form of contacting bodies, has *some* dependency of the normal force F_N on the indentation depth d, with the function $F_N(d)$ being unambiguous and monotonous. It is obvious that it is *always* possible to formally construct a one-dimensional profile in contact with an elastic foundation that has the same dependency $F_N(d)$ such that the two normal contact problems are equivalent *with respect to force–displacement relations*. The same will be valid for all other problems that can be reduced to the normal contact problem.

2.4 Class V

Normal contact problem of *arbitrary* bodies (including randomly or non-randomly rough) also belongs to the realm of applicability of the MDR *as long as only the force–displacement relations are of interest* in constructing the 1D models. Corresponding proof for the case of tangential contact with a constant coefficient of friction has been done in [45].

The restriction to the force–displacement relations is important and is stated in all fundamental works on the MDR at least since 2012. For example, in the introduction to the book on the method of dimensionality reduction [1] it is clearly stated that "Not all problems involving rough surfaces can be solved with the reduction method, but only those that deal with forces and relative displacements, such as problems dealing with contact stiffness, electrical or heat conduction, and frictional force. The area of application is limited but very large and includes many problems which have meaningful implications in engineering".[2]

3 Papers Criticized in the Persson's Paper

In the paper [32], it is stated that "In a recent series of papers, Popov et al. [3–23] have used the 1D mapping procedure to study different contact mechanics problems

[1] Sometimes confusion occurs about the notion of "1D mapping". We speak about a 1D model, if the degrees of freedom fill a one-dimensional space (as in the case of a linear elastic foundation). In an elastic half-space, on the contrary, the degrees of freedom fill a three-dimensional space. We speak therefore about a "reduction from 3D to 1D". Of course, 1D does not mean that the model has only one degree of freedom. Typically we run models with several thousand up to millions degrees of freedom of the "1D model".

[2] Note that a more precise statement should read: "... contact stiffness and (electric or heat) constriction resistance". This follows from Barber's theorem [42] about the contact stiffness and, of course, does not account for many physical effects such as the presence of surface films.

(contact stiffness, contact area and rubber friction) between randomly rough surfaces". Before discussing the essence of the criticism by Persson, we have to take a closer look at these 21 references.

1. In three papers [6], [8], and [14], which do not use and *do not even mention the method of dimensionality reduction*, only direct three-dimensional simulations by the method of boundary elements are utilized.
2. Papers [23] and [21] are devoted to description of micro-robots driven by frictional contacts of a steel plate and ruby balls. This falls into the region of exact validity of the MDR (see Case I in Sect. 2) and is in no way related to the MDR-based model of contact of rough surfaces.
3. In paper [15], axis-symmetric contacts are considered in the case of heterogeneous media. It is clearly stated that this is an approximate procedure and its quality is discussed in context of other works on the contact of heterogeneous elastic bodies.
4. Papers [5], [9], and [19] are indeed devoted to the *contact stiffness* of self-affine fractal surfaces. As explained in Sect. 1, the pure force–displacement relations—and the problem of *contact stiffness* fall into this category—belong to the realm of exact applicability of MDR. The *existence* of an equivalent one-dimensional profile is a trivial consequence of the monotony of the force–displacement relation. Because of the independence of the springs in the one-dimensional system, the number of different profiles having the same normal contact properties is actually infinite. The problem is thus not the *existence* of an equivalent profile, but its practical construction. In the mentioned papers, we suggested a rule for determining the equivalent profile based on the early work of Geike [7]. We then have shown that the stiffness can be described well by the MDR-based model if the mentioned generation rule (so-called modified rule of Geike) for the profile is used. This possibility was not merely *assumed*, but verified by comparison of the MDR results with complete three-dimensional numerical solutions with the method of boundary elements. Moreover, it has been *analytically* shown that the behaviour of the contact stiffness is correct in the limit of both very small and very large normal forces. In the paper [5], this approach has been extended to self-affine surfaces with Hurst's exponents in the region between -1 and 3. The quality of this description of the contact stiffness can be seen from Fig. 1 and 2 in our previous reply [43] to Persson's critique. While our rule of producing the equivalent profile is analytically substantiated in the limits of very small and very large normal forces, it remains an empirical rule. We are completely aware of the fact that these results are not exact, but just a good approximation. We describe this as follows [43]: "The 1D and 3D results coincide for very low forces and for complete contact. In the crossover region, the results for the H in the vicinity of 2/3 (which is the most relevant value for many technical surfaces) coincide too, while there is a moderate deviation for very small ($H \approx 0$) end very large ($H \approx 2$) values of H".

Thus, the first part of the papers cited by Persson has nothing to do with MDR, and the second part falls into the realm of exact validity of the MDR. In the last fourth group of papers, the MDR is not *used to study the contact mechanics of rough surfaces*, as stated by Persson, but rather, analytical considerations and three-dimensional numerical solutions are used to *validate* predictions of the simple MDR-based model obtained via application of the 1D mapping procedure (which in this case is *approximate*). To all papers mentioned in this section, the criticism brought forward by Persson is *not applicable.*

4 What do MDR-Based Models Fail to Describe in Mechanics of Rough Contact?

In the statement cited at the beginning of Sect. 2, three areas are mentioned, for which the 1D mapping procedure has been used: *contact stiffness, contact area, and rubber friction* between randomly rough surfaces. However, in the case of rough surfaces, not all these problems can be modelled with the 1D mapping procedure equally well. For example, already in the introduction to the book [1] one can read: "The mapping of real contact area remains *beyond* the realm of application. The method of dimensionality reduction is able to map contact areas for very small initial stage of indentation but not in a general case". In the review paper [33], there is a section "Objections to the method of reduction of dimensionality and its area of application". We read there: "That the contact area and the contact stiffness cannot be both described correctly by the reduction method follows immediately from the well-known property that the saturation value of the contact stiffness is achieved in macroscopic systems much earlier than the complete material contact (see the book [11], §7.3).

There is a simple rule to judge whether some physical property can or cannot be described by a MDR-based model: Properties that scale *linearly* with the size of the indenter (e.g. the contact stiffness) are described either "exactly" (if the correct mapping rule is applied) or very well (if a simplified, approximate rule is applied). Quantities that have other scaling, in particular the contact area, *cannot* be described with the MDR-based models developed so far. This is *a well-known* fact published by the authors of MDR much earlier than the criticism by Persson. At the early stages of development of the MDR, it was suggested [7] that the method can describe the contact area. According to the present understanding, this claim was erroneous, even though the numerical data presented in [7] were correct. The reason for this discrepancy is the very small grid resolution (64 × 64 points) of 3D simulations that were available to us at that time. For this size of the system, the range of correct description of the contact area for very small forces is about 10–20 % of the apparent area. For larger contacts, the 1D simulation result deviates very rapidly from the correct three-dimensional result. This was realized and published before any external criticism of the MDR.

Of course, it is clear without saying that the MDR is a method of solution of contact mechanical problems for continuous bodies and is applicable only under general conditions of applicability of continuum mechanics.

5 MDR and Elastomer Friction

Let us now turn to the analysis of the main part of the paper by Persson, where he claims to consider friction of a smooth elastomer moving relatively to a rough rigid base.

Let me start with the general statement that the problem of elastomer friction *does not* belong to the realm of *exact* validity of the MDR. In our papers on elastomer friction, we therefore underline that we speak about *qualitative* analysis ("We do not claim that the reported results can be directly applied for the friction of a true three-dimensional elastomer" [16].). However, I will argue that the MDR can be used for qualitative analysis of elastomer friction in practical situations and that the arguments of Persson are erroneous.

In reality in his paper Persson *does not* consider *friction* (that is tangential sliding), but merely the static normal contact. He argues that the complete contact in the three-dimensional systems is achieved at much higher forces than the complete contact in the one-dimensional system. This statement is *equivalent* to the statement that the contact area achieves saturation at larger forces than the contact stiffness. This conclusion is trivial and well known, while the consideration of Persson does not add anything new to the discussions of the topic which can be found in our works or in former Persson's comments on MDR. However, this statement *firstly* does not say anything about the force of friction. Thus, the sadden "conclusion" of Persson: "As a result, the rubber friction force, as obtained using the 1D mapping procedure suggested by Popov et al., is bound to exhibit a very different pressure or load dependence than in a correct 3D treatment" is completely non-motivated and not justified. And *secondly*, the whole logics presented by Persson become incorrect as soon as we really consider *tangential* movement. In this case, the forces in the *relevant* region of velocities (in the vicinity of the plateau of the dependency of the force of friction on velocity) will be mostly determined not by the roughness as in the normal contact but by the surface gradients (due to viscosity of the elastomer). And the gradients—contrary to the roughness—are larger at smaller scale. Thus, the statement that only a small force will be needed to achieve a complete contact in the one-dimensional case compared with the three-dimensional case will not be valid any more. Persson then puts the question whether it is possible to improve the procedure to become a qualitative agreement and suggests that for this purpose one has to use a profile which "exhibits short-wavelength roughness with an amplitude that increases rather than decreases when the wavelength of the roughness component decreases". But this is exactly the property which is already present in the MDR—but not for roughness which does not play an important role in elastomer friction, but for gradients!

In the interest of the readers of "Tribology Letters", it is necessary to describe in the following the true logics behind the applications of the MDR to the elastomer friction in a more detail. The basic idea of the following consideration is that the elastomer friction can also be reduced to the normal contact problem—however only approximately ("qualitatively") and under some restrictions.

Let me start with consideration of a single contact. In this case, there exist a very popular estimation of the coefficient of friction which can be found, e.g. in the paper by Persson and Tosatti [46] or in the book [47]. This estimation states that the coefficient of friction is a product of the typical surface gradient ∇z with a rheological factor: $\mu \approx \nabla z \cdot \mathrm{Im}(G(\omega))/|G(\omega)|$, where ω is a "typical frequency" which can be estimated as the size of the contact divided by the sliding velocity. In Chapter 11 of the book [1], it is shown that exactly the same estimation is valid also for a contact of a body of the same size with the one-dimensional elastic foundation defined according to the rules of the MDR. Thus, the above estimation is independent on the dimensionality of the system and is—other than often believed—not related to any interaction of the degrees of freedom. The reason for the robustness of this estimation is very simple: It comes from the estimation of the energy losses of a contact at some average frequency and this is given correctly by the MDR as the force–displacement relations are reproduced correctly by the MDR. Let us recollect that in the context of elastomer friction "correctly" means: with the overall precision of the above estimation, this means "qualitatively correct".

Naturally, the characteristic frequency depends on the size of contact and thus will depend on the indentation depth. But the interrelation of the indentation depth and the size of the contact—in first approximation—does not depend on the rheology at all—this is the basic result of Radok. (In the second approximation, it of course does depend on velocity and rheology but never too strong; a detailed analysis based on numerical BEM-solutions is done in [49].) Thus, the above estimation for the coefficient of friction will be qualitatively correct also for any given indentation depth. And finally, as the indentation depth is related to the normal force via stiffness—which is correctly described by the MDR (in the case of elastomers again in the sense of estimation, i.e. approximately)—this will be valid also for the dependence of the coefficient of friction on the normal force. For pure viscous media, Kürschner [48] has shown this in his recent paper.

Let me summarize the reason for the validity of the above-mentioned estimation: This estimation reformulates the problem of elastomer friction as a problem of a periodic normal contact with a characteristic frequency determined by sliding velocity and topography. In doing so, it places the problem of elastomer friction in the class of problems which can be “qualitatively reduced” to the normal contact problem. And this allows to approximately apply the MDR to this class of contacts. The main source of *only approximate validity* of the above estimation in application to rough surfaces is that the “characteristic frequency” (used in this mapping) cannot be defined unambiguously for arbitrary rough surfaces. However, as argued in the paper [50], for typical fractal topographies there exists a characteristic scale determining friction. Normally this is the “smallest relevant scale” which is determined by the upper cut-off wave vector (which in turn is determined by some physical factors which may be not known and whose real physical nature is not really important). This “governing scale” provides the characteristic frequency needed for the application of the MDR to elastomer friction. Additionally to the mere existence of the governing scale, the characteristic gradients should be equal on this scale in the three-dimensional and one-dimensional models. But the rule of Geike [7] for the spectrum transformation guarantees the invariance of the rms roughness, rms slope, and rms curvature, thus guaranteeing the equivalence of the parameter on the decisive “governing scale”. From this analysis, it follows, of course, that the MDR will work qualitatively well only in the vicinity of the plateau of the dependence of the coefficient of friction on the sliding velocity (which only is the region of any practical importance in the majority of practical cases). Thus, the analysis of friction in the state of complete contact (which Persson claims to carry out but does not do this) is simply irrelevant for this application.

We would like to stress again that the tangential contact of rough viscoelastic materials does not belong to the realm of exact validity of the MDR, so we have always to be careful in describing the MDR model and the conclusions drawn from it. This approach can be followed in all papers of the series under consideration that are devoted to elastomer friction (see, in particular, [3, 4, 16]).

6 Conclusions

It is shown that the paper [32] by Persson contains multiple, let us say, *inaccurately* interpreted or incomplete citations of the works he criticizes or uses for his support. Most of the criticized papers have nothing to do with the MDR or are dealing with axially symmetric contacts. Even the papers in which—as Persson claims—“it has been argued… that this model fails even qualitatively to describe the contact mechanics correctly [24–27]” are cited incorrectly: Both the third (Pastewka et al. [26]) and the forth (Scaraggi et al. [27]) papers in the list do not contain any comparison with MDR. The first two papers are comments of Persson with essentially identical criticism which already has been answered in [43].

The main part of the paper [32] is claimed to be devoted to the elastomer friction. However, Persson considers instead the static normal contact and repeats then his old criticism, which was already put forward in his two previous comments on the MDR. The argument is basically that in a rough contact the saturation of the contact area is achieved at larger forces than the saturation of contact stiffness. To conclude from this that the “contact mechanics of rough surfaces”, in particular the contact stiffness, cannot be described with MDR, is wrong. On the contrary, in a series of papers we presented the unambiguous validation of applicability of the MDR for describing contact stiffness through comparison with direct three-dimensional simulations (see in particular [5] and Section. 10.7 in [1]).

In conclusion, we summarize briefly the main claims of the MDR and its regions of applicability according to the current understanding:

- MDR provides exact solutions for axis-symmetric Hertz-type contact problems, and this is not only for the normal contact with elastic bodies but also for tangential contact with a constant coefficient of friction under arbitrary loading histories in the Cattaneo–Mindlin approximation and for normal contact with arbitrary elastomer with linear rheology as well as for adhesive contacts in the JKR approximation.
- MDR provides exact solutions for the normal and tangential contact with elastic and viscoelastic bodies also for arbitrary topographies *but now* (*for arbitrary*

topographies) *only in the sense of force–displacement relations* $F_N(d)$, provided the normal contact problem is solved (and thus the exact equivalent profile is known). For the case of self-affine randomly rough surfaces, we have derived an empirical rule for the equivalent profile, which was validated by direct three-dimensional simulations. In other cases, the general solution of the normal contact with an elastic half-space is a prerequisite for the MDR application.

- MDR provides a qualitative description of elastomer friction in the region of the plateau of the coefficient of friction as function of velocity. We do not claim that this part of MDR applications is exact, so here one has to be cautious. We note that generally speaking, the MDR-based models of rough contact (depending on the accuracy and efficiency of the equivalent profile for the rough surfaces in contact) may fail to capture some details that depend on the short-wavelength roughness. Still, there are no exact three-dimensional results in the relevant region of the plateau of the coefficient of friction, which could be used for reference. Thus, this area of current research remains under investigation.

References

1. Popov, V.L., Heß, M.: Method of Dimensionality Reduction in Contact Mechanics and Friction. Springer, Berlin (2015)
2. Sneddon, I.N.: The relation between load and penetration in the axisymmetric Boussinesq problem for a punch of arbitrary profile. Int. J. Eng. Sci. **3**, 47 (1965)
3. Li, Q., Dimaki, A., Popov, M., Psakhie, S.G., Popov, V.L.: Kinetics of the coefficient of friction of elastomers. Sci. Rep. **4**, 5795 (2014)
4. Popov, V.L., Voll, L., Li, Q., Chai, Y.S., Popov, M.: Generalized law of friction between elastomers and differently shaped rough bodies. Sci. Rep. **4**, 3750 (2014)
5. Pohrt, R., Popov, V.L.: Contact stiffness of randomly rough surfaces. Sci. Rep. **3**, 3293 (2013)
6. Paggi, M., Pohrt, R., Popov, V.L.: Partial-slip frictional response of rough surfaces. Sci. Rep. **4**, 5178 (2014)
7. Geike, T., Popov, V.L.: Mapping of three-dimensional contact problems into one dimension. Phys. Rev. E **76**, 036710 (2007)
8. Pohrt, R., Popov, V.L.: Normal contact stiffness of elastic solids with fractal rough surfaces. Phys. Rev. Lett. **108**, 104301 (2012)
9. Pohrt, R., Popov, V.L., Filippov, A.: Normal contact stiffness of elastic solids with fractal rough surfaces for one- and three-dimensional systems. Phys. Rev. E **86**, 026710 (2012)
10. Popov, V.L., Filippov, A.E.: Force of friction between fractal rough surface and elastomer. Tech. Phys. Lett. **36**, 525 (2010)
11. Popov, V.L.: Contact Mechanics and Friction, Physical Principles and Applications. Springer, Berlin (2010). **(See Chapter 19)**
12. Phys. Mesomech. 15, 4 (2012)
13. Popov, V.L., Filippov, A.E.: Applicability of a reduced model to description of real contacts between rough surfaces with different Hurst exponents. Tech. Phys. Lett. **34**, 722 (2008)
14. Grzemba, B., Pohrt, R., Teidelt, E., et al.: Maximum micro-slip in tangential contact of randomly rough self-affine surfaces. Wear **309**, 256 (2014)
15. Popov, V.L.: Method of dimensionality reduction in contact mechanics and tribology. Heterogeneous media. Phys. Mesomech. **17**, 50 (2014)
16. Li, Q., Popov, M., Dimaki, A., et al.: Friction between a viscoelastic body and a rigid surface with random self-affine roughness. Phys. Rev. Lett. **111**, 034301 (2013)
17. Psakhie, S.G., Popov, V.L.: Mesoscopic nature of friction and numerical simulation methods in tribology. Phys. Mesomech. **15**, 251 (2012)
18. Dimaki, A.V., Popov, V.L.: The method of reduction of dimensionality and its application to simulation of elastomer friction under complex dynamic loads. Phys. Mesomech. **15**, 319 (2012)
19. Pohrt, R., Popov, V.L.: Investigation of the dry normal contact between fractal rough surfaces using the reduction method, comparison to 3D simulations. Phys. Mesomech. **15**, 275 (2012)
20. Popov, V.L., Filippov, A.E.: Adhesive properties of contacts between elastic bodies with randomly rough self-affine surfaces: a simulation with the method of reduction of dimensionality. Phys. Mesomech. **15**, 324 (2012)
21. Nguyen, H.X., Teidelt, E., Popov, V.L., et al.: Dynamic tangential contact of rough surfaces in stick-slip microdrives: modeling and validation using the method of dimensionality reduction. Phys. Mesomech. **17**(4), 304–310 (2014)
22. Popov, V.L.: Basic ideas and applications of the method of reduction of dimensionality in contact mechanics. Phys. Mesomech. **15**(5–6), 254–263 (2012)
23. Teidelt, E., Willert, E., Filippov, A.E., et al.: Modeling of the dynamic contact in stick-slip microdrives using the method of reduction of dimensionality. Phys. Mesomech. **15**, 287 (2012)
24. Lyashenko, I.A., Pastewka, L., Persson, B.N.J.: On the validity of the method of reduction of dimensionality: area of contact, average interfacial separation and contact stiffness. Tribol. Lett. **52**(3), 223–229 (2013)
25. Lyashenko, I.A., Pastewka, L., Persson, B.N.J.: Comment on friction between a viscoelastic body and a rigid surface with random self-affine roughness. Phys. Rev. Lett. **111**, 189401 (2013)
26. Pastewka, L., Prodanov, N., Lorenz, B., Müser, M.H., Robbins, M.O., Persson, B.N.J.: Finite-size effects in the interfacial stiffness of rough elastic contacts. Phys. Rev. E **87**, 062809 (2013). See also arXiv:1210.4255v1 and arXiv:1210.4635v1
27. Scaraggi, M., Putignano, C., Carbone, G.: Elastic contact of rough surfaces: a simple criterion to make 2D isotropic roughness equivalent to 1D one. Wear **297**, 811 (2013)
28. Persson, B.N.J.: Theory of rubber friction and contact mechanics. J. Chem. Phys. **115**, 3840 (2001)
29. Klüppel, M., Heinrich, G.: Rubber friction on self-affine road tracks. Rubber Chem. Technol. **73**, 578 (2000)
30. Westergaard, H.M.: Bearing pressures and cracks. Trans. ASME J. Appl. Mech. **6**, 49 (1939)
31. Persson, B.N.J.: On the fractal dimension of rough surfaces. Tribol. Lett. **54**, 99 (2014)
32. Persson, B.N.J.: Contact mechanics for randomly rough surfaces: on the validity of the method of reduction of dimensionality. Tribol. Lett. (2015). doi:10.1007/s11249-015-0498-1
33. Popov, V.L.: Method of reduction of dimensionality in contact and friction mechanics: a linkage between micro and macro scales. Friction **1**(1), 41–62 (2013)
34. Cattaneo, C.: Sul contatto di due corpi elastici: distribuzione locale degli sforzi. Rendiconti dell'Accademia nazionale dei Lincei **27**, 342–348, 434–436, 474–478 (1938)
35. Mindlin, R.D.: Compliance of elastic bodies in contact. J. Appl. Mech. **16**, 259–268 (1949)
36. Jaeger, J.: Axi-symmetric bodies of equal material in contact under torsion or shift. Arch. Appl. Mech. **65**, 478–487 (1995)
37. Ciavarella, M.: Tangential loading of general three-dimensional contacts. J. Appl. Mech. **65**, 998–1003 (1998)

38. Borodich, F.M., Galanov, B.A.: Molecular adhesive contact for indenters of nonideal shapes. In: ICTAM04, Abstracts Book and CD-Rom Proceedings, Warsaw, IPPT PAN (2004)
39. Yao, H., Gao, H.: Optimal shapes for adhesive binding between two elastic bodies. J. Colloid Interface Sci. **298**(2), 564–572 (2006)
40. Lee, E.H.: Stress analysis in viscoelastic bodies. Quart. Appl. Math. **13**, 183–190 (1955)
41. Radok, J.R.M.: Viscoelastic stress analysis. Quart. Appl. Math. **15**, 198–202 (1957)
42. Barber, J.R.: Bounds on the electrical resistance between contacting elastic rough bodies. Proc. R. Soc. Lond. Ser. A **459**, 53–66 (2003)
43. Li, Q., Popov, M., Dimaki, A., et al.: Li et al. Reply. Phys. Rev. Lett. **111**, 189402 (2013)
44. Argatov, I.I., Popov, V.L.: Rebound indentation problem for a viscoelastic half-space and axisymmetric indenter—solution by the method of dimensionality reduction. arXiv: arxiv.org/abs/1508.04237 (2015)
45. Popov, V.L, Pohrt, R., Hess, M.: General procedure for solution of contact problems under dynamic normal and tangential loading based on the known solution of normal contact problem. arXiv: arxiv.org/abs/1508.04242 (2015)
46. Persson, B.N.J., Tosatti, E.: Qualitative theory of rubber friction and wear. J. Chem. Phys. **112**, 2021–2029 (2000)
47. Popov, V.L.: Contact Mechanics and Friction. Physical Principles and Application. Springer, Berlin (2010)
48. Kürschner, S.: Friction between a rigid body and a model elastomer having a linear viscous rheology. ZAMM **95**, 822–830 (2015)
49. Kusche S.: Frictional force between a rotationally symmetric indenter and a viscoelastic half-space. ZAMM (2015) (submitted)
50. Popov, V.L., Dimaki, A., Psakhie, S., Popov, M.: On the role of scales in contact mechanics and friction between elastomers and randomly rough self-affine surfaces. Sci. Rep. **5**, 11139 (2015)
51. Heß, M.: Über die Abbildung ausgewählter dreidimensionaler Kontakte auf Systeme mit niedrigerer räumlicher Dimension. Cuvillier-Verlag, Göttingen (2011)
52. Aleshin, V., Matar, O.B., Van Den Abeele, K.: Method of memory diagrams for mechanical frictional contacts subject to arbitrary 2D loading. Int. J. Solids Struct. **60**, 84–95 (2015)
53. Ting, T.C.T.: The contact stresses between a rigid indenter and a viscoelastic half-space. J. Appl. Mech. **33**, 845–854 (1966)