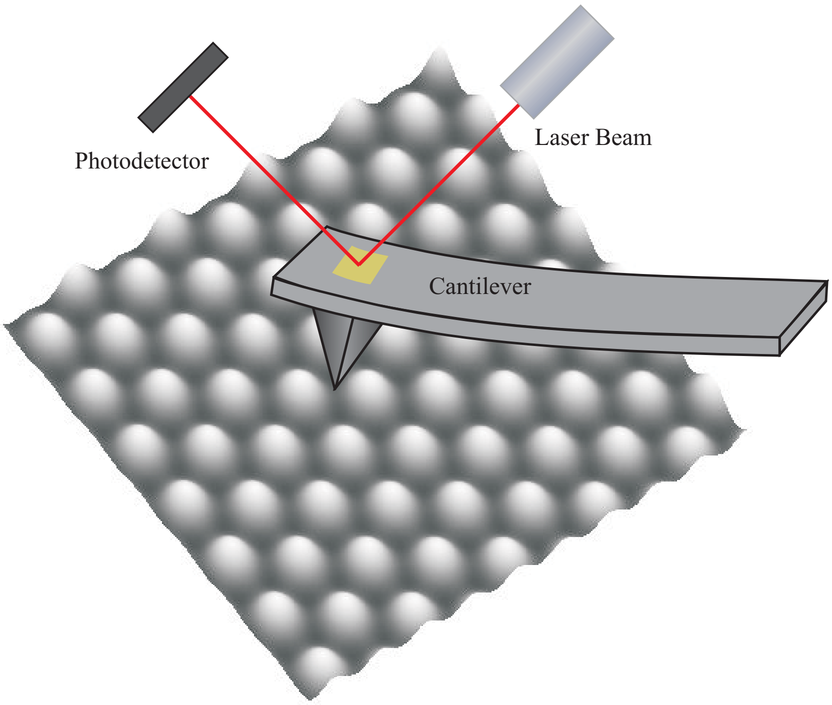
11 The Prandtl-Tomlinson Model for Dry Friction



11.1 Introduction

The development of experimental methods to investigate frictional processes on the atomic scale and numerical simulation methods has heralded in a drastic growth in the number of studies in the area of friction of solid bodies on the atomic scale. A simple model, known as the "Tomlinson model," can be used as the basis for many investigations of frictional mechanisms on the atomic scale. It was suggested by Prandtl in 1928 to describe the plastic deformations in crystals[1]. The paper by Tomlinson[2] that is often cited in this context did not contain the model today known as the "Tomlinson model" and suggests an adhesive contribution to friction. In the following, we will call this model the "Prandtl-Tomlinson

[1] L. Prandtl, Ein Gedankenmodell zur kinetischen Theorie der festen Körper. ZAMM, 1928, Vol. 8, p. 85-106.

[2] G.A. Tomlinson, A molecular theory of friction. The London, Edinburgh, and Dublin philosophical magazine and journal of science, 1929, Vol. 7 (46 Supplement), p. 905.).

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model." Prandtl considered the one dimensional movement of a point mass in a periodic potential with the wave number k being acted upon by an external force and being damped proportional to velocity[3] (Fig. 11.1):

$$m\ddot{x} = F - \eta\dot{x} - N\sin kx\ . \tag{11.1}$$

Here, x is the coordinate of the body, m its mass, F the external force acting upon it, η the damping coefficient, N the amplitude of the periodic force, and k the wave number.

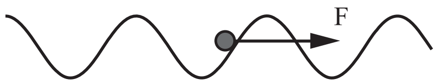

Fig. 11.1 Prandtl-Tomlinson model: A point mass in a periodic potential.

The model from Prandtl-Tomlinson describes many fundamental properties of dry friction. Actually, we must apply a minimum force to the body so that a macroscopic movement can even begin. This minimum force is none other than the macroscopic force of static friction. If the body is in motion and the force reduced, then the body will generally continue to move, even with a smaller force than the force of static friction, because it already possesses a part of the necessary energy due to its inertia. Macroscopically, this means that the kinetic friction can be smaller than the static friction, which is a frequently recurring characteristic of dry friction. The force of static friction in the model in Fig. 11.1 is equal to N .

The success of the model, variations and generalizations of which are investigated in innumerable publications and are drawn on to interpret many tribological processes, is due to the fact that it is a simplistic model that accounts for two of the most important fundamental properties of an arbitrary frictional system. It describes a body being acted upon by a periodic conservative force with an average value of zero in combination with a dissipating force which is proportional to velocity. Without the conservative force, no static friction can exist. Without damping, no macroscopic sliding frictional force can exist. These two essential properties are present in the Prandtl-Tomlinson model. In this sense, the Prandtl-Tomlinson model is the simplest usable model of a tribological system. Essentially, the Prandtl-Tomlinson model is a restatement and further simplification of the view of Coulomb about the "interlocking" of surfaces as the origin of friction.

Obviously, the model cannot represent all of the subtleties of a real tribological system. For instance, there is no change of the surface potential caused by wear in this model. It should be noted, however, that it is fundamentally possible to expand the model to take plastic deformations into account. In this context, it should be mentioned once more that the model from L. Prandtl in 1928 was proposed precisely to describe plastic deformations in crystals.

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[3] In this way, for example, the movement of the tip of an atomic force microscope over a crystalline surface can be described.

In this chapter, we investigate the Prandtl-Tomlinson model as well as several applications and generalizations of it.

11.2 Basic Properties of the Prandtl-Tomlinson Model

If a body is at rest and a force F is applied to it, then its equilibrium position moves to the point x, which satisfies the equation

$$F = N \sin kx . \tag{11.2}$$

This equation has a solution only when $F \le N$. So, the force of static friction, in this model, is equal to

$$F_s = N . \tag{11.3}$$

For a larger force, no equilibrium is possible and the body enters into macroscopic motion[4]. In this model, every macroscopic movement of the body, from a microscopic point of view, is a superposition of a constant speed and a periodic oscillation, as is shown in Fig. 11.2 a. In this figure, the results of the numerical integration of Equation (11.1) are presented. The tangential force changes slowly from zero to some maximum value larger than the force of static friction and then decreases afterwards. The curve shows the instantaneous speed as a function of the instantaneous force. After the critical force is reached, the body begins to move with a finite velocity. If the force is decreased, the body can continue to move when acted upon by forces smaller than the force of static friction. At a specific critical velocity, the macroscopic movement stops, the body oscillates about a potential minimum, and then comes to a standstill.

On the macroscopic scale, we do not perceive the microscopic oscillations. The movement described above is described from a macroscopic point of view as a quasi-stationary frictional process. The dependence of the average velocity on the applied force is perceived as the *macroscopic law of friction* to a macroscopic observer (Fig. 11.2 b).

[4] Here, we call "macroscopic" the movement of a body in a spatial domain much larger than the potential period. Conversely, we call the length scale smaller than, or comparable to, the wavelength of the potential "microscopic."

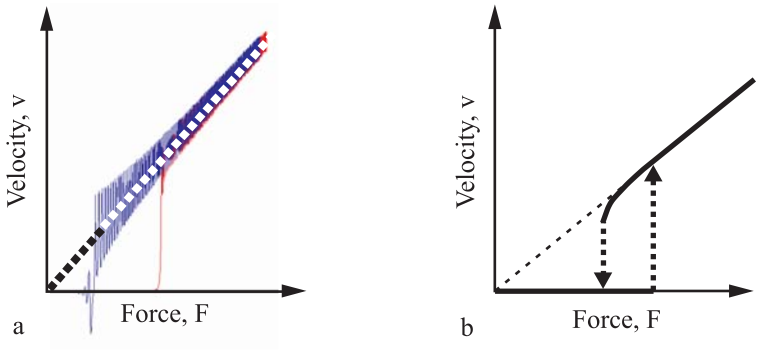

Fig. 11.2 (a) Dependence of instantaneous velocity on the force (increasing linearly with time) in the Prandtl-Tomlinson model. (b) Macroscopic law of friction: dependence of the average velocity on the force.

Case of small damping
If the damping is $\eta = 0$ and the body is put into motion, then it will continue to move indefinitely, even in the absence of an external force ($F = 0$). Thereby, conservation of energy is valid

$$E_0 = \frac{mv^2}{2} - \frac{N}{k}\cos kx = const, \quad \text{for} \quad \eta = 0, \quad F = 0\,. \tag{11.4}$$

In this case, the resulting velocity as a function of the x-coordinate is

$$v = \sqrt{\frac{2}{m}\left(E_0 + \frac{N}{k}\cos kx\right)}, \quad \text{for} \quad \eta = 0, \quad F = 0\,. \tag{11.5}$$

In the presence of a small damping, a small force must be applied in order to maintain periodic motion. The motion is periodic if the work performed by an external force F over a period of $a = 2\pi / k$ is equal to the energy loss, $\int_0^T \eta v^2(t)dt$:

$$\frac{2\pi F}{k} = \int_0^T \eta v^2(t)dt = \int_0^a \eta v(x)dx = \eta \int_0^a \sqrt{\frac{2}{m}\left(E_0 + \frac{N}{k}\cos kx\right)}dx\,. \tag{11.6}$$

The smallest force F_1 at which a macroscopic movement still exists is given by (11.6) with $E_0 = N / k$:

$$\frac{F_1}{N} = \frac{4}{\pi}\frac{\eta}{\sqrt{mkN}}\,. \tag{11.7}$$

The damping at which the force of kinetic friction would be equal to the force of static friction has the order of magnitude of

$$\frac{\eta}{\sqrt{mkN}} \approx 1 \tag{11.8}$$

and indicates the boundary between the considered case with little damping (*under-damped* system) and the case with much damping (*over-damped* system).

Case of large damping
For large damping, one can neglect the inertia term in (11.1):

$$0 = F - \eta\dot{x} - N\sin kx\ . \tag{11.9}$$

This is known as *over-damped* motion. The equation of motion in this case is a first-order differential equation. It can be written in the form

$$\dot{x} = \frac{dx}{dt} = \frac{F}{\eta} - \frac{N}{\eta}\sin kx\ . \tag{11.10}$$

One spatial period is traversed in the time

$$T = \int_0^{2\pi/k} \frac{dx}{\frac{F}{\eta} - \frac{N}{\eta}\sin kx} = \frac{\eta}{kN}\int_0^{2\pi} \frac{dz}{\frac{F}{N} - \sin z} = \frac{\eta}{kN}\frac{2\pi}{\sqrt{\left(\frac{F}{N}\right)^2 - 1}}\ . \tag{11.11}$$

The average speed is, therefore,

$$\bar{v} = \frac{a}{T} = \frac{\sqrt{F^2 - N^2}}{\eta}\ . \tag{11.12}$$

The force, as a function of the average speed, is

$$F = \sqrt{N^2 + (\eta\bar{v})^2}\ . \tag{11.13}$$

This dependence is presented in Fig. 11.3.

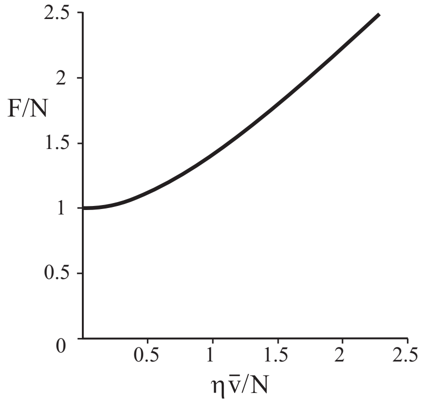

Fig. 11.3 The law of friction for the Prandtl-Tomlinson model in the over-damped case.

The "phase diagram" for the Prandtl-Tomlinson model
In order to investigate the properties of the Prandtl-Tomlinson model for arbitrary parameters, we will introduce the dimensionless variables into (11.1)

$$x = \xi \tilde{x}, \quad t = \tau \tilde{t} . \tag{11.14}$$

From this, the equation of motion assumes the form

$$\frac{m\xi}{N}\frac{\tilde{x}''}{\tau^2} = \frac{F}{N} - \frac{\eta\xi}{N}\frac{\tilde{x}'}{\tau} - \sin(k\xi\tilde{x}) , \tag{11.15}$$

where the prime indicates the derivative $\partial / \partial \tilde{t}$. We choose

$$k\xi = 1, \quad \tau^2 \frac{N}{m\xi} = 1 \tag{11.16}$$

and write Equation (11.15) in the form

$$\tilde{x}'' + \frac{\eta}{\sqrt{mkN}}\tilde{x}' + \sin\tilde{x} = \frac{F}{N} . \tag{11.17}$$

It now only contains two dimensionless parameters,

$$\kappa_1 = \frac{\eta}{\sqrt{mkN}} , \quad \kappa_2 = \frac{F}{N} . \tag{11.18}$$

In Fig. 11.4, the "phase portrait" of the system is presented. The nature of the motion in the dimensionless coordinates is dependent only on the region in which the system lies on the parameter plane (κ_1, κ_2) in this figure.

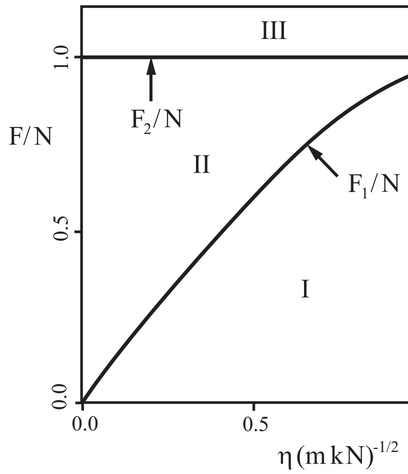

Fig. 11.4 Two critical forces, F_1 and F_2, as a functions of the damping coefficient.

For $\kappa_1 = \dfrac{\eta}{\sqrt{mkN}} < 1.193$, there are three force domains (shown in Fig. 11.4 as I, II and III), which are separated by the critical forces F_1 and F_2. For $F > F_2$, there is no equilibrium solution and the body moves unrestricted. If the force decreases, the body comes to a standstill when $F < F_1$. Between the domains in which only the static state ($F < F_1$) or only motion ($F > F_2$) exist, there is a domain of *bistability*, in which the body can exist in a static state or a state of motion, depending on its initial condition. This domain of bistability does not exist if the damping is larger than a critical value:

$$\frac{\eta}{\sqrt{mkN}} > 1.193\,. \tag{11.19}$$

For small damping coefficients, the critical force F_1 is given by (11.7).

11.3 Elastic Instability

The simplest generalization of the Prandtl-Tomlinson model is presented in Fig. 11.5. Instead of being acted upon by a constant force, the body is connected to a spring (stiffness c) which is fastened to a sliding sled that moves in the horizontal direction. This model is more suited to describe the movement of the tip of an atomic force microscope than the original model from Prandtl-Tomlinson, because it takes into account the stiffness of the arm of the microscope in the simplest manner.

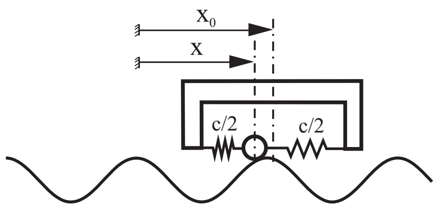

Fig. 11.5 A body in a periodic potential pulled with a spring.

In this case, the equation of motion is

$$m\ddot{x} + \eta \dot{x} + \frac{\partial U}{\partial x} = c(x_0 - x) . \tag{11.20}$$

If the sled is pulled slowly with a constant velocity, then the body can be found at every point in time at an equilibrium position $x(x_0)$, where $x_0 = x_0(t)$ is the coordinate of the sled. The average value of the spring force is equal to the (macroscopic) force of friction. If there is only one equilibrium position $x(x_0)$ for *every* x_0, then the average value of the force acting on the body is *exactly zero*. In order to show that, we investigate the total potential energy of the body:

$$U_{tot}(x, x_0) = U(x) + \frac{1}{2} c \left(x - x_0 \right)^2 . \tag{11.21}$$

The equilibrium position is determined from the condition

$$U'_{tot}(x, x_0) = U'(x) + c\left(x - x_0\right) = 0 , \tag{11.22}$$

where the prime indicates the derivative $\partial / \partial x$. The average value of this force (over time, which in this case, is the same as over x_0) is equal to

$$\bar{F}_{base} = -\frac{1}{L} \int_0^L U' dx_0 . \tag{11.23}$$

Here, L is the spatial period of the potential. If we differentiate Equation (11.22), we obtain

$$\left(U''(x) + c\right) dx = c dx_0 . \tag{11.24}$$

With this, one can replace the integration of (11.23) over dx_0 with integration over dx:

$$\bar{F}_{base} = -\frac{1}{L} \int_0^L U' \left(1 + \frac{U''}{c} \right) dx = -\frac{1}{L} \left[U(x) + \frac{U'^2(x)}{2c} \right]_0^L = 0 . \tag{11.25}$$

The resulting average force is zero, because both $U(x)$ and $U'^2(x)$ are periodic functions of x. From this, it follows that *the force of friction under these conditions is exactly zero*. This is valid for arbitrarily defined periodic potentials.

The situation changes considerably if the equilibrium coordinate x is a non-continuous function of x_0, so that in some points, Equation (11.24) is not satisfied. As an example, we will investigate the system shown in Fig. 11.5 with a potential of the form

$$U(x) = -\frac{N}{k}\cos kx \,. \tag{11.26}$$

The equilibrium condition (11.22) takes the form

$$-\sin kx = \frac{c}{N}\left(x - x_0\right). \tag{11.27}$$

The functions $-\sin kx$ and $\frac{c}{N}\left(x - x_0\right)$ are shown in Fig. 11.6 for various x_0. Their intersection indicates the equilibrium coordinate of the body. If $c/Nk > 1$, then x is a continuous function of the coordinate x_0 of the sled, which is illustrated through example calculations in Fig. 11.6 b with $c/Nk = 1.5$. If, on the other hand, the stiffness of the spring is smaller than a critical value:

$$c/Nk < 1 \,, \tag{11.28}$$

then the dependence of the equilibrium coordinate on x_0 has jump discontinuities (Fig. 11.6 d). In this case, the time-averaged force is not equal to zero. The dependence of the spring force on the coordinate x_0 for the case of a weak spring ($c/Nk = 0.1$) is presented in Fig. 11.7.

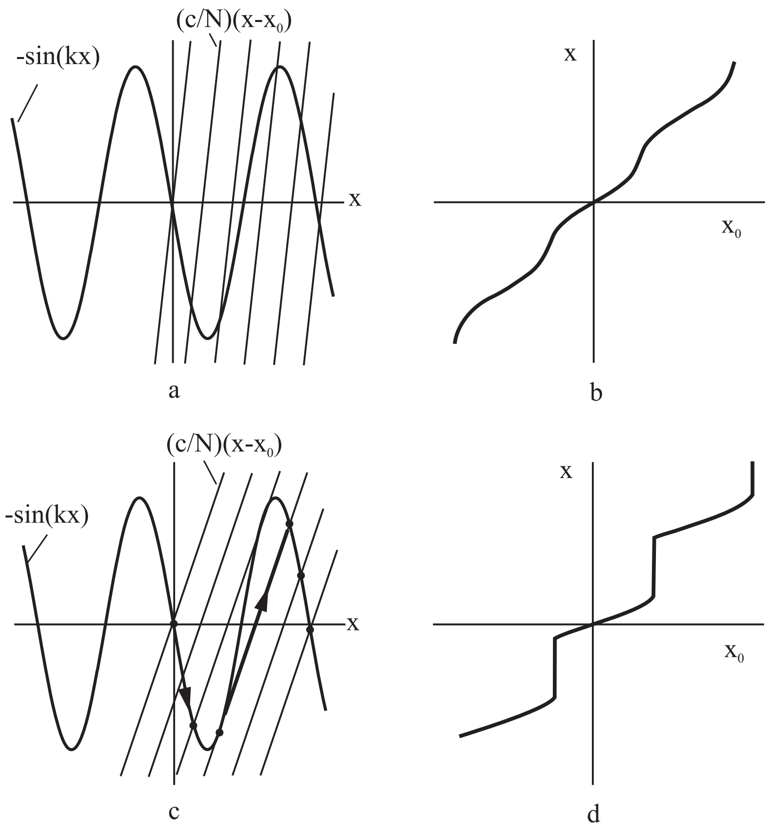

Fig. 11.6 The functions $-\sin kx$ and the linear function $\frac{c}{N}\left(x-x_0\right)$ are plotted in (a) for $c/Nk=1.5$ and in (c) for $c/Nk=0.5$. If x_0 increases, the linear function shifts to the right. The equilibrium coordinate is a continuous function of x_0, when $c/Nk>1$ (b) and has jump discontinuities when $c/Nk<1$ (d).

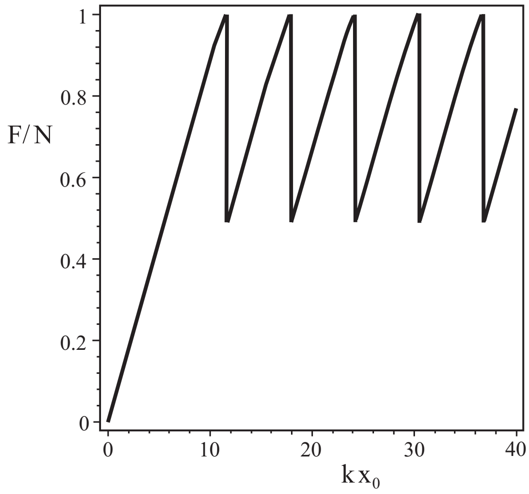

Fig. 11.7 Dependence of the force acting on the body in model (11.20) as a function of the coordinate of the sliding sled x_0 for the case $c / Nk = 0.1$. Because of the emergence of elastic instabilities, the average value of the force is not equal to zero.

11.4 Superlubricity

Experimental and theoretical investigations in recent years have led to the conclusion that in an "atomically close" contact between two crystalline solid bodies, it is possible to have no friction, provided that the periods of the crystal lattices are incompatible (as shown in Fig. 11.8). An additional requirement is that no elastic instabilities may appear in the contact between both bodies. The cause for the absence of static friction is that the atoms of one of the crystal lattices are placed in all possible relative energy states in relation to the other lattice. Therefore, the movement of the body leads merely to another distribution of the atoms that sit in the positions of low and high energy, but it causes no change in the average (macroscopic) energy of the body. Because of this, even an infinitesimally small force can put the body into motion.

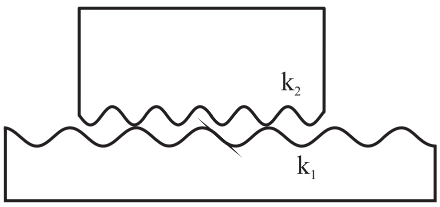

Fig. 11.8 Contact between two periodic surfaces (e.g. two crystals) with different lattice periods.

These considerations are, of course, independent of the scale. They are valid for the contact between two macroscopic structured surfaces, for example, between corrugated rubber base and a corrugated steel plate. *As long as the periods of the structures of both surfaces are different and no elastic instabilities occur, the structures give no contribution to static friction.*

11.5 Nanomachines: Concepts for Micro and Nano-Actuators

Because of the tendency to miniaturize mechanical devices, one must ask oneself the question of what the theoretical limit of miniaturization is. An important aspect thereby is whether or not it is possible to transfer thermal or chemical potential energy into the energy of directional movement even on the smallest, atomic scales. Many researchers have taken the motion of the so-called motor protein along periodically built microfibers as an example for many investigations on nano-machines. All motor proteins have a similar structure consisting of two “heads” and a connecting element. The length of the connection can be changed through the burning of “energy molecules.” By heating the protein molecule, it transforms from a globular state to that of a random coil, whereby the length of the bond increases. After cooling, the bond again takes its original length.

Most of the methods to produce directed motion of microscopic or molecular objects which are discussed in literature are based on the interactions between a moving object and a heterogeneous, and in most cases periodic, substrate. The driven object can consist of one or more bodies whose separation distances are controllable. The underlying substrate can be symmetric as well as asymmetric. For non-symmetric bases, one uses the “ratchet-and-pawl” principle[5]. A directional movement, however, is also possible in symmetric potentials.

In this section, we will illustrate the idea behind nano-machines using the example of a “three-body machine.” From a mathematical point of view, we are dealing with the movement of a multi-body system in a (spatially) periodic potential which is a simple generalization of the Prandtl-Tomlinson model.

Below, we show how controlling the length of the connections between the bodies in a periodic potential can lead to a directional movement of the system for which the movement direction as well as speed are arbitrarily controllable.

Singular points and bifurcation sets of a three-body machine

We consider a mass point in a periodic potential (Fig. 11.9) that is connected by two massless rods of lengths l_1 and l_2. The potential energy of the system is equal to

$$U = U_0 \left(\cos(k(x - l_1)) + \cos(kx) + \cos(k(x + l_2)) \right), \tag{11.29}$$

[5] These machines are often just called “ratchets.”

where $k = 2\pi / a$ is the wave number and a is the spatial period of the potential. The potential energy can be rewritten in the form

$$U = U_0 \sqrt{\left(\sin kl_1 - \sin kl_2\right)^2 + \left(1 + \cos kl_1 + \cos kl_2\right)^2} \cos(kx - \varphi) , \qquad (11.30)$$

where

$$\tan \varphi = \frac{\sin kl_1 - \sin kl_2}{1 + \cos kl_1 + \cos kl_2} . \qquad (11.31)$$

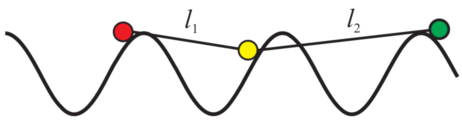

Fig. 11.9 Three-body machine.

The phase φ is a continuous function of the parameters l_1 and l_2 over an arbitrary path on the parameter plane (l_1, l_2), as long as this path does not cross any *singular points*, in which the amplitude of the potential (11.30) approaches zero and the phase (11.31) is undefined. The position of these points is determined by the conditions

$$\sin kl_1 - \sin kl_2 = 0 \qquad (11.32)$$

and

$$1 + \cos kl_1 + \cos kl_2 = 0 . \qquad (11.33)$$

From this, it follows that

$$kl_1 = \pi \pm \pi / 3 + 2\pi n , \qquad kl_2 = \pi \pm \pi / 3 + 2\pi m , \qquad (11.34)$$

where m and n are integers. The position of singular points on the (l_1, l_2)-plane is shown in Fig. 11.10. All of these points can be obtained by periodically repeating the two points $(kl_1, kl_2) = \left(2\pi / 3, 2\pi / 3\right)$ and $(kl_1, kl_2) = \left(4\pi / 3, 4\pi / 3\right)$ as multiples of 2π .

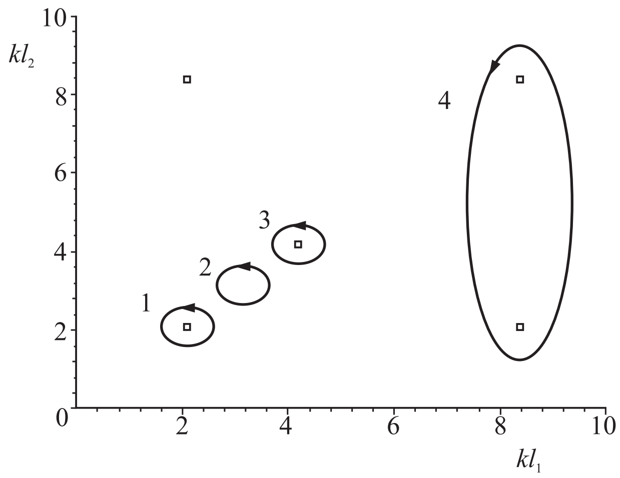

Fig. 11.10 Positions of singular points of a “three-body machine.”

Conditions for controlled directional motion

Now, we assume that the lengths l_1 and l_2 are arbitrarily controllable. When they change so that one singular point in Fig. 11.10 is encircled by a closed path (path 1), then the phase is decreased by 2π. By closing a loop about a second point (path 3), it is increased by 2π. We assign the first point the topological index -1 and the second point the index +1. Generally, the phases of a closed-path on the (l_1, l_2)-plane change by $2\pi i$, where i is the sum of the indices of all of the points enclosed by the path. Path 2 in Fig. 11.10, for example, surrounds no singular point. Therefore, the phase does not change along the path. Path 4 surrounds two points with the index -1 and, therefore, the phase changes by -4π over the complete path. A phase change of 2π corresponds to the movement of the three-body machine over one spatial period.

A periodic change in the rod lengths l_1 and l_2 on a path which encloses singular points with the non-zero sum of the topological indices leads to directional motion of the system. If a path in the (l_1, l_2)-plane is traversed at an angular frequency of ω, then the system would move at the macroscopic (average) velocity $v = \frac{\omega i}{k}$.

An interesting question is if this “machine” can move even when being opposed by an external force and, therefore, be used to carry loads. In order to answer this question, we allow an external force of $-F$ to act on the system. This leads to an additional term Fx in the potential energy so that the total potential energy assumes the form

$$U_{tot} = U_0 \left(\cos(k(x-l_1)) + \cos(kx) + \cos(k(x+l_2)) \right) + Fx \,. \tag{11.35}$$

We determine the bifurcation set (also called the “catastrophe set”) for this potential. The bifurcation set is understood to be the parameter set for which the number of equilibrium points of the potential changes and, therefore, the equilibrium position is generally no longer a continuous function of the parameters l_1 and l_2. It is determined by two conditions:

$$\frac{\partial U_{tot}}{\partial x} = 0 \tag{11.36}$$

and

$$\frac{\partial^2 U_{tot}}{\partial x^2} = 0 \,. \tag{11.37}$$

The first condition means that we are dealing with an equilibrium position. The second condition indicates that it is exactly the moment in which the equilibrium loses its stability. In our case, (11.36) yields

$$\frac{\partial U_{tot}}{\partial x} = U_0 k \left(-\sin(k(x-l_1)) - \sin(kx) - \sin(k(x+l_2)) \right) + F = 0 \tag{11.38}$$

and (11.37)

$$\frac{\partial^2 U_{tot}}{\partial x^2} = U_0 k^2 \left(-\cos(k(x-l_1)) - \cos(kx) - \cos(k(x+l_2)) \right) = 0 \,. \tag{11.39}$$

By applying the addition theorems of trigonometry and subsequently summing the squares, this equation can be written in the form

$$\left(1 + \cos kl_1 + \cos kl_2\right)^2 + (\sin kl_1 - \sin kl_2)^2 = \left(F / U_0 k\right)^2 . \tag{11.40}$$

The bifurcation set determined by this equation is shown in Fig. 11.11 for 4 different values of the parameter $f = F / U_0 k$. A translational movement is induced when the lengths l_1 and l_2 vary over a closed path that completely surrounds a closed bifurcation set so that the phase in every point remains a continuous function of l_1 and l_2. This is obviously only possible for $f < 1$. The maximum driving force is, therefore, equal to $F_{\max} = U_0 k$.

With certain special variations in time of the lengths of l_1 and l_2, the directional motion of the system can be especially clearly seen. By choosing

$$l_1 = (4/3)\pi / k + l_0 \cos(\omega t) \text{ and } l_2 = (4/3)\pi / k + l_0 \cos(\omega t + \varphi) \tag{11.41}$$

with

$$\varphi = (2/3)\pi \tag{11.42}$$

and $l_0 \ll 1/k$, the potential energy (11.29) takes the following form:

$$\begin{aligned} & U_0 k l_0 \left[\sin(kx+\pi/3)\cos(\omega t+2\pi/3)-\sin(kx-\pi/3)\cos\omega t\right] \\ & = U_0 k l_0 \left(\sqrt{3}/2\right)\cos(kx+\omega t+\pi/3) \ . \end{aligned} \tag{11.43}$$

This is a periodic profile that propagates at a constant velocity ω/k in the negative x-direction. The system moves together with the potential wave in one of its minima.

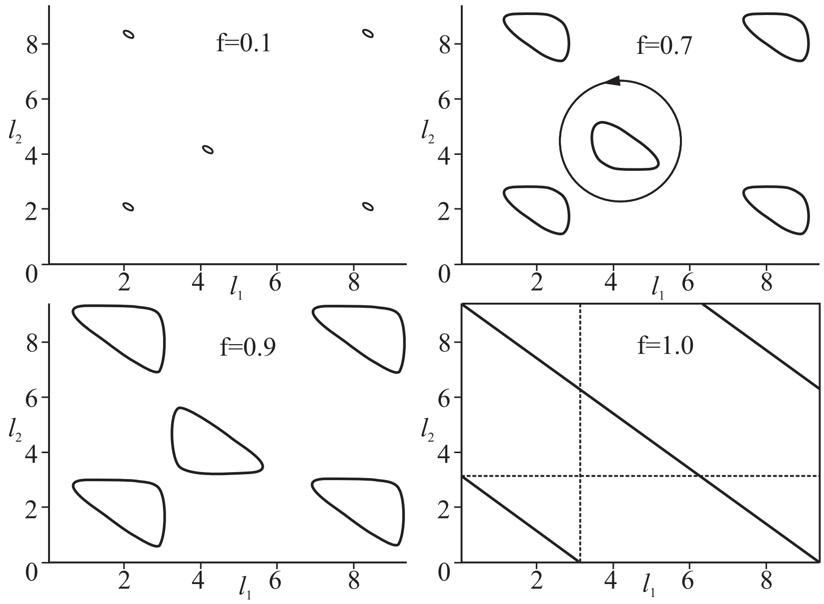

Abb. 11.11 Bifurcation sets of the potential (11.35) for various external forces $f = F/U_0k$. Directional motion is possible as long as the bifurcation sets form closed forms that can be surrounded by a path without it intersecting them.

The ideas discussed in this section are used actively in nano-tribology in order to describe, among others things, the molecular motors in cells, muscular contraction, and the design of nano-motors.

Problems

Problem 1: Investigate a somewhat modified Prandtl-Tomlinson model: a point mass m moves under the applied force F in a periodic potential that is formed by repeating the domain of a parabola shown below (Fig. 11.12):

$$U(x) = \frac{1}{2}cx^2 \text{ for } -\frac{a}{2} \le x \le \frac{a}{2}$$

with

$$U(x+a) = U(x).$$

Furthermore, there is damping proportional to velocity with the damping coefficient η. Determine: (a) the force of static friction, (b) the minimum velocity at which macroscopic movement ceases, (c) the force of kinetic friction as a function of the average sliding velocity and damping, and (d) the "phase diagram" of a system similar to the classical Prandtl-Tomlinson model.

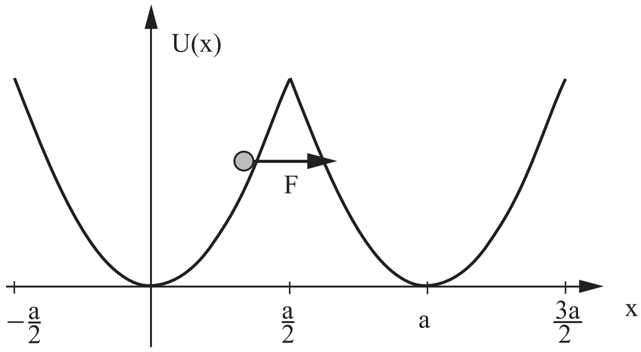

Fig. 11.12 Modified Prandtl-Tomlinson model with parabolic potential.

Solution: The force of static friction is equal to the maximum slope of the potential, which is reached at the end of the period (e.g. at $x = a/2$):

$$F_s = \frac{ca}{2}.$$

The equation of motion within the period of the potential is

$$m\ddot{x} + \eta\dot{x} + cx = F.$$

The minimum force at which macroscopic movement still exists obviously corresponds to the situation at which the body starts moving (at $x = -a/2$ with a velocity $\dot{x} = 0$) and stops (at $x = a/2$ with the velocity $\dot{x} = 0$ again). This is exactly half of one damped oscillation period in a parabolic potential. The vibration frequency of damped oscillations is generally known to be equal to

$$\omega^* = \sqrt{\omega_0^2 - \delta^2}$$

with $\omega_0^2 = c/m$ and $\delta = \eta/2m$. According to this, a spatial period of the potential is repeated after the time

$$T = \frac{\pi}{\omega^*}.$$

The smallest average velocity possible for a steady state unbounded movement is equal to

$$v_{\min} = \frac{a}{T} = \frac{a\omega^*}{\pi} = \frac{a}{\pi}\sqrt{\frac{c}{m} - \left(\frac{\eta}{2m}\right)^2}.$$

The minimal force at which macroscopic movement is still possible can be established most easily by using the following considerations. The total potential energy of the body, taking into account the external force F, is equal to

$$U = \frac{cx^2}{2} - Fx = \frac{c}{2}\left[\left(x - \frac{F}{c}\right)^2 - \left(\frac{F}{c}\right)^2\right].$$

The change in the potential energy between the point $x = -a/2$ and the minimum potential energy is $\Delta U_0 = \frac{c}{2}\left(\frac{a}{2} + \frac{F}{c}\right)^2$, and the change in the potential energy between the minimum and the point $x = a/2$ is $\Delta U_1 = -\frac{c}{2}\left(\frac{a}{2} - \frac{F}{c}\right)^2$. At the minimum force, the body traverses exactly half of the damped oscillation period: from $-a/2$ to $a/2$. It is known from vibration theory that the energy of a damped oscillation decreases exponentially according to $e^{-2\delta t}$. The ratio of the aforementioned energies is, therefore, $e^{-2\delta T}$:

$$\left(\frac{a - 2F/c}{a + 2F/c}\right)^2 = e^{-2\delta T}.$$

From this, it follows that

$$F = \frac{ac}{2}\frac{1 - e^{-\delta T}}{1 + e^{-\delta T}} = F_s \frac{1 - e^{-\delta T}}{1 + e^{-\delta T}}$$

with

$$\delta T = \frac{\pi\eta}{\sqrt{4mc - \eta^2}} = \frac{\pi}{\sqrt{\frac{4mc}{\eta^2} - 1}}.$$

The dependence of the normalized force F/F_s on the dimensionless parameter $\eta/\sqrt{4mc}$ is shown in Fig. 11.13.

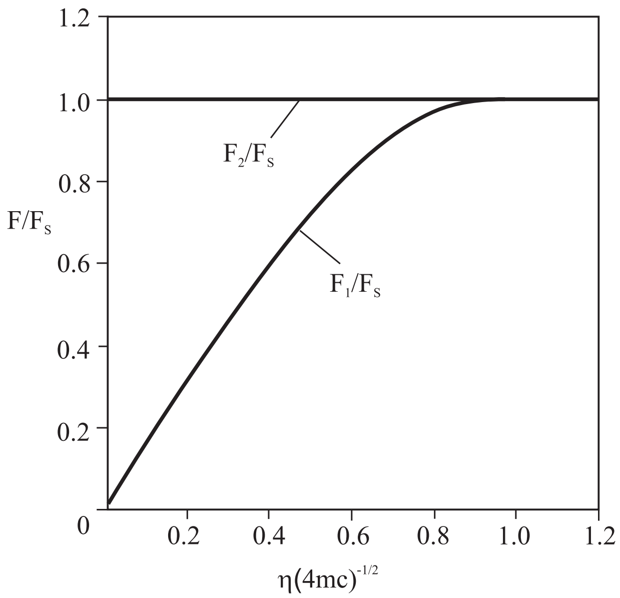

Fig. 11.13 Phase diagram of a modified Prandtl-Tomlinson model with parabolic sections with two critical forces F_1 and F_2.

Problem 2: A point mass is coupled with a rigid slide, by means of springs which have a "vertical stiffness" $c_\perp$ and a "tangential stiffness" $c_{\parallel}$ [6]. It is placed on a sinusoidal profile ($y = h_0 \cos kx$) as shown in Fig. 11.14. Then, the slide moves to the right. Determine the conditions required for elastic instabilities to occur in this system.

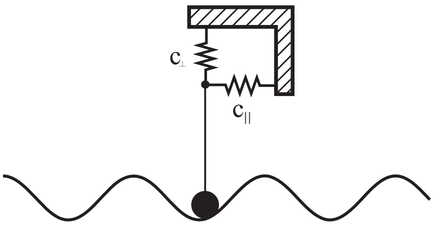

Fig. 11.14 A body that is elastically coupled in the vertical and horizontal directions sliding on a corrugated surface.

Solution: The potential energy of the system is equal to

$$U(x, y, x_0, y_0) = \frac{c_\perp}{2}(y - y_0)^2 + \frac{c_{\parallel}}{2}(x - x_0)^2 .$$

[6] This model can describe, for example, an element of the elastic profile of a rubber sole.

For the system described in the problem statement, the relationships $y = h_0 \cos kx$ and $y_0 = -h_0$ are valid. The potential energy assumes the form

$$U(x, x_0) = \frac{c_\perp}{2}\left(h_0 \cos kx + h_0\right)^2 + \frac{c_\parallel}{2}\left(x - x_0\right)^2 .$$

The condition for an elastic instability to occur is

$$\frac{\partial^2 U}{\partial x^2} = -c_\perp h_0^2 k^2 \left[\cos kx + \cos 2kx\right] + c_\parallel = 0 .$$

This equation has solutions and, therefore, the system exhibits instabilities when

$$c_\parallel < 2 c_\perp h_0^2 k^2 .$$