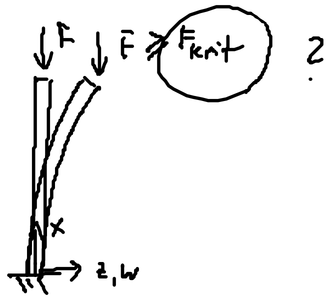
15. Übung – Knicken



DGL für Knicken:

$$\left(EI\,w''(x)\right)'' = -F\,w''(x)$$

$$EI\,w^{IV}(x) + F\,w''(x) = 0 \quad \Big|\, : EI$$

$$w^{IV}(x) + \underbrace{\frac{F}{EI}}_{\lambda^2}\,w''(x) = 0$$

$$\boxed{w^{IV}(x) + \lambda^2 w''(x) = 0} \quad (1)$$

allgemeine Lösung:

$$w(x) = A\cos(\lambda x) + B\sin(\lambda x) + \underset{(Cx)}{C\lambda x} + D \quad (2)$$

$$w'(x) = -\lambda A\sin(\lambda x) + \lambda B\cos(\lambda x) + C\lambda$$

$$w''(x) = -\lambda^2 A\,\underline{\cos(\lambda x)} - \lambda^2 B\,\underline{\sin(\lambda x)}$$

$$w'''(x) = \lambda^3 A\sin(\lambda x) - \lambda^3 B\cos(\lambda x)$$

$$w^{IV}(x) = \lambda^4 A\,\underline{\cos(\lambda x)} + \lambda^4 \underline{B\sin(\lambda x)} = \lambda^2\left(\lambda^2 A\cos(\lambda x) + \lambda^2 B\sin(\lambda x)\right)$$

$$w^{IV}(x) = -\lambda^2 w''(x) \quad \Big|\, + \lambda^2 w''(x)$$

$$w^{IV}(x) + \lambda^2 w''(x) = 0$$

Vorgehen:

- 4 RBen bestimmen, FS in ausgelenkter Lage!
- RBen einsetzen
- Matrix aufstellen $\underline{K}\begin{bmatrix}A\\B\\C\\D\end{bmatrix} = \vec{0}$
- $\det(\underline{K}) \overset{!}{=} 0 \Rightarrow$ Eigenwertgleichung
- kleinstes $\lambda \neq 0$ ablesen

$\Rightarrow F_{krit} = EI\,\lambda_1^2$

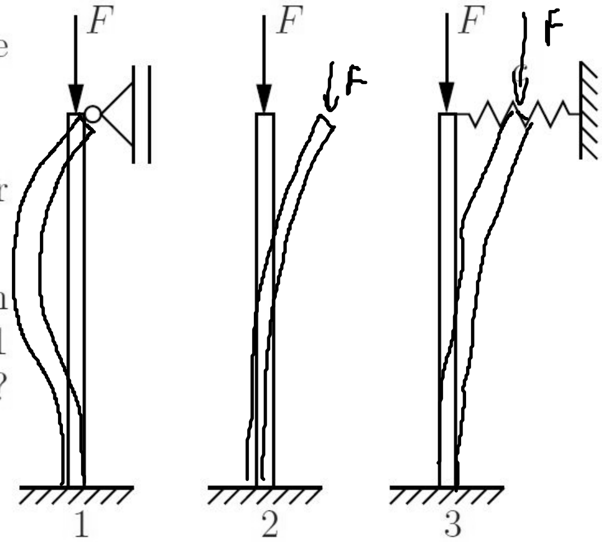
$\lambda^2 = \frac{F}{EI}$

148. Ein Balken der Biegesteifigkeit EI und der Länge l werde auf verschiedene Arten gelagert.

(a) Berechnen Sie die kritischen Lasten F_{krit} für die Varianten 1 bis 3!

(b) Wie müsste das Verhältnis der Balkenlängen l_1/l_2 gewählt werden, damit für Variante 1 und 2 dieselbe kritische Last berechnet wird?

Geg.: F, c, l, EI



a) Man erkennt:

- Fall 1 entspricht Fall 3 mit $c \to \infty$
- Fall 2 entspricht Fall 3 mit $c = 0$

$\Rightarrow$ Fall 3 lösen, $c = 0$, $c \to \infty$ einsetzen!

1) DGL:

$$(EI w'')'' + F w'' = 0$$

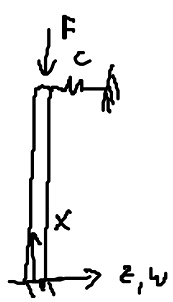
$$EI w^{IV} + F w'' = 0 \quad | : EI$$

$$w^{IV} + \lambda^2 w'' = 0, \quad \text{mit } \lambda^2 = \frac{F}{EI}$$

2) allg. Lös.:

$$w(x) = A\cos(\lambda x) + B\sin(\lambda x) + C\lambda x + D$$

3) RBen:

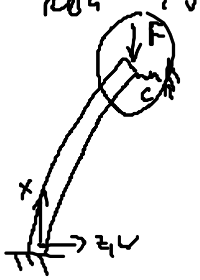


$w(x=0) = 0$ RB1

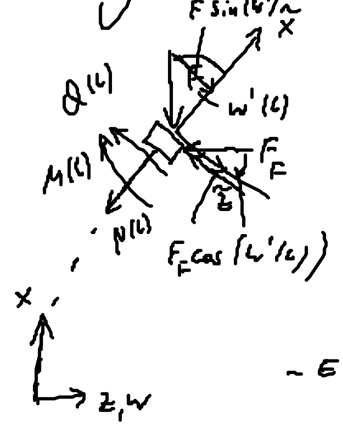
$w'(x=0) = 0$ RB2

$M(x=l) = 0$ RB3

für RB4 FS in ausgelenkter Lage $(x = l)$:



FS:



GGB:

$$\tilde{z}: \; 0 = -Q(l) + F\sin(w'(l)) - F_F\cos(w'(l))$$

$$\underbrace{\cancel{Q(l)}}_{-EI w'''(l)} = F\,\overset{w'(l)}{\cancel{\sin(w'(l))}} - F_F \underbrace{\cancel{\cos(w'(l))}}_{\approx 1}$$

mit: $Q(l) = -EI w'''(l)$

$\underbrace{\sin(\alpha) \approx \alpha}_{\alpha \ll 1}, \quad \cos(\alpha) \approx 1$

$$\left.\right\} \; -EI w'''(l) = F w'(l) - c\,w(l) \qquad \text{RB4}$$

$F_F = c \cdot \Delta s = c \cdot w(l)$, $\quad \Delta s = w(l)$

4) Einsetzen:

$$w(x) = A\cos(\lambda x) + B\sin(\lambda x) + C\lambda x + D$$
$$w'(x) = -\lambda A\sin(\lambda x) + \lambda B\cos(\lambda x) + C\lambda$$
$$w''(x) = -\lambda^2 A\cos(\lambda x) - \lambda^2 B\sin(\lambda x)$$
$$w'''(x) = \lambda^3 A\sin(\lambda x) - \lambda^3 B\cos(\lambda x)$$
$$w''''(x) = \lambda^4 A\cos(\lambda x) + \lambda^4 B\sin(\lambda x)$$

RB1: $w(0) = 0 = A + 0 + 0 + D$

$D = -A \quad (1)$

RB2: $w'(0) = 0 = \lambda B + C\lambda$

$C\lambda = -\lambda B \quad (2)$

RB3: $M(l) = 0 \Rightarrow -\cancel{EI} w''(l) = 0 = -\lambda^2 A\cos(\lambda l) - \lambda^2 B\sin(\lambda l) \quad (3)$

RB4: $EI w'''(l) + F w'(l) - c\,w(l) = 0$

$$\cancel{EI}\left(\lambda^3 A\sin(\lambda l) - \lambda^3 B\cos(\lambda l)\right) + \overset{\lambda^2}{\frac{F}{EI}}\left(-\lambda A\sin(\lambda l) + \lambda B\cos(\lambda l) + \underbrace{\cancel{C\lambda}}_{-\lambda B\,(2)}\right)$$
$$- \frac{c}{EI}\left(A\cos(\lambda l) + B\sin(\lambda l) - \lambda B l + \underbrace{\cancel{D}}_{-A\,(1)}\right) = 0 \quad (4) \qquad \Big| :EI$$

$$\lambda^3\left(\underline{\cancel{A\sin(\lambda l)}} - \underline{\cancel{B\cos(\lambda l)}}\right) + \lambda^3\left(\underline{-\cancel{A\sin(\lambda l)}} + \underline{\cancel{B\cos(\lambda l)}} - B\right)$$
$$- \frac{c}{EI}\left(A\cos(\lambda l) + B\sin(\lambda l) - \lambda B l - A\right) = 0$$

$$-\lambda^3 B - \frac{c}{EI}\Big(\underbrace{\cancel{A\cos(\lambda l) + B\sin(\lambda l)}}_{(3)'} - \lambda B l - A\Big) = 0 \qquad (4)'$$

$$-\cancel{\lambda^2} A\cos(\lambda l) - \cancel{\lambda^2} B\sin(\lambda l) = 0 \qquad \Big|:\lambda^2$$

$$+A\,\underline{\cos(\lambda l)} + B\,\underline{\sin(\lambda l)} = 0 \qquad (3)'$$

$$-\lambda^3 B - \frac{c}{EI}\left(-\lambda B l - A\right) = 0$$

$$\underline{\frac{c}{EI}}\,A + \underline{\left(\frac{c}{EI}\lambda l - \lambda^3\right)}\,B = 0 \qquad (4)'$$

5) Matrix:

$$(3)',(4)' \Longrightarrow \underbrace{\begin{bmatrix} \cos(\lambda l) & \sin(\lambda l) \\ \frac{c}{EI} & \frac{c}{EI}\lambda l - \lambda^3 \end{bmatrix}}_{\underline{K}} \begin{pmatrix} A \\ B \end{pmatrix} = \vec{0}$$

Es gibt nur dann nicht-triviale Lösung wenn $\det(\underline{K}) = 0$

$$\det(\underline{K}) = \cos(\lambda l)\left(\frac{c}{EI}\lambda l - \lambda^3\right) - \frac{c}{EI}\sin(\lambda l) = 0 \qquad \Big| + \frac{c}{EI}\sin(\lambda l)$$

$$\cos(\lambda l)\left(\frac{c}{EI}\lambda l - \lambda^3\right) = \frac{c}{EI}\sin(\lambda l) \qquad \Big| \cdot \frac{EI}{c}\ ;\ :\cos(\lambda l)$$

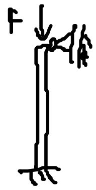
$$\lambda l - \lambda^3 \frac{EI}{c} = \tan(\lambda l)$$

$$\boxed{\lambda l\left(1 - (\lambda l)^2 \frac{EI}{l^3 c}\right) = \tan(\lambda l)} \qquad (5)$$

Eigenwertgleichung!

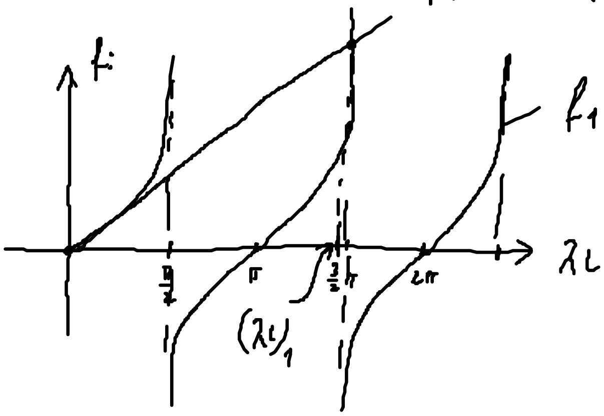
6.) F_{krit} bestimmen

Fall 1:



$c \rightarrow \infty$ in (5):

$$\underbrace{\tan(\lambda L)}_{f_1} = \underbrace{\lambda L}_{f_2}$$



Ablesen: $(\lambda L)_1 \approx 4{,}5 \Rightarrow \lambda_1 = \frac{4{,}5}{L}$

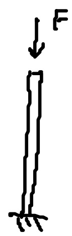
$\uparrow$

kleinster Eigenwert $\lambda L \neq 0$

damit: $F_{krit} = EI(\lambda_1)^2$

$$F_{krit}^1 = \frac{EI}{L^2}(4{,}5)^2$$

Fall 2:

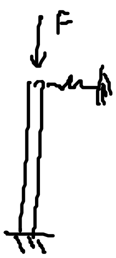


$c = 0$ in (5)

$$\tan(\lambda L) = \lim_{c \to 0} \left(\lambda L \left(1 - (\lambda L)^2 \frac{EI}{cL^3}\right)\right) = -\infty$$

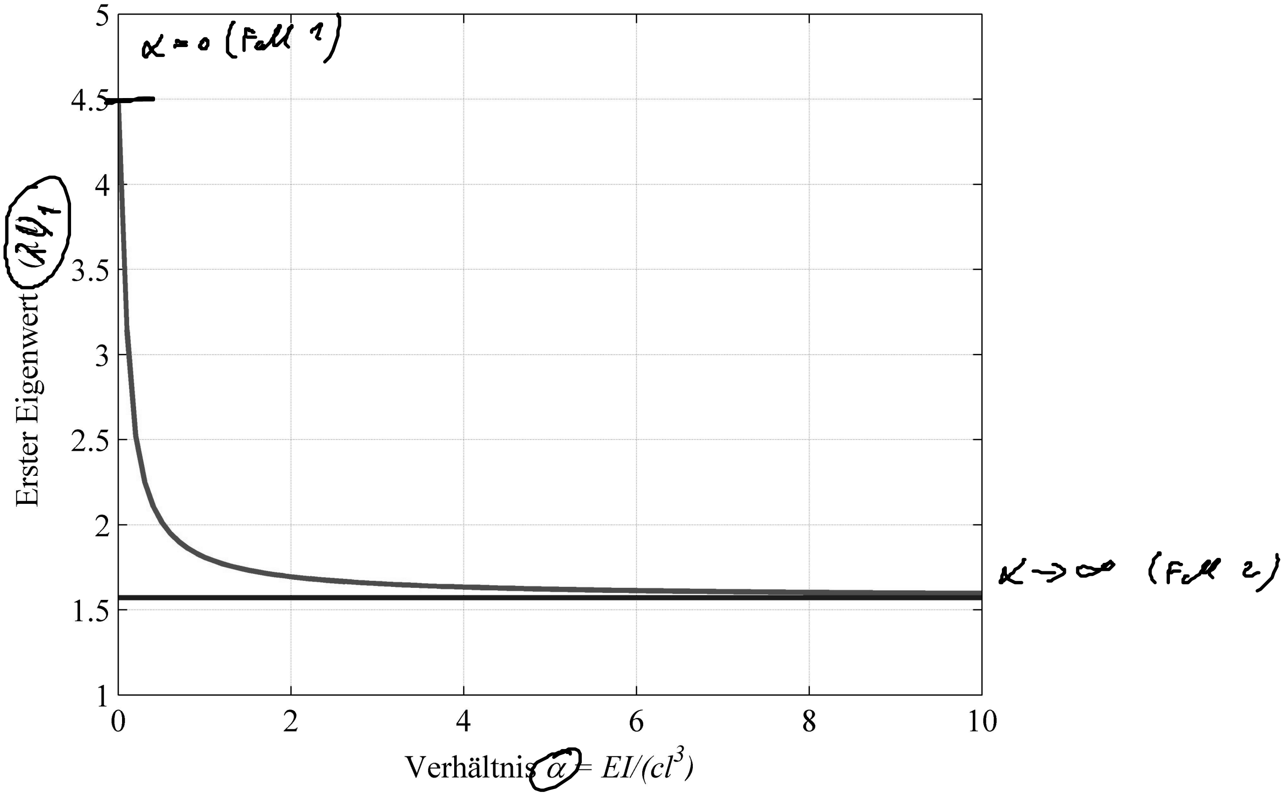
damit: $(\lambda L)_1 = \frac{\pi}{2} \Longrightarrow F_{krit}^2 = \frac{EI}{L^2}\left(\frac{\pi}{2}\right)^2$

Fall 3:



$$\tan(\lambda L) = \lambda L\left(1 - (\lambda L)^2 \underbrace{\frac{EI}{cL^3}}_{\alpha}\right)$$

$$\tan(\lambda L) = \lambda L\left(1 - \alpha (\lambda L)^2\right) \Longrightarrow \text{Numerisch lösen}$$



Tut: 155, 153; Ha: 148, 154

Experiment:

Euler-Fall I:

$$F_{krit}^{?} = \frac{EI}{4}\frac{\pi^2}{l^2} = \frac{1}{4}\, 210\cdot 10^9 \frac{N}{m^2}\cdot\frac{1}{12}\cdot 10\,mm\cdot(1\,mm)^3\cdot 10^{-12}\frac{m^4}{mm^4}\cdot\pi^2\frac{1}{(1\,m)}$$

$$\approx 0{,}4\,N \quad\Rightarrow\quad m_{krit}\approx 40\,g$$